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Handouts of the Course on “Probability in Boolean and in Many-valued
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To my students in Africa, Europe, and South-America

Preface

In his 1931 paper “Sul significato soggettivo della probabilità”, published in *Fundamenta Mathematicae*, de Finetti introduced the notion of a consistent betting system and characterized it by a consistency theorem, also known as de Finetti’s Dutch book theorem. On §16, page 328, he summarized his results as follows (italics by de Finetti):

Dimostrare le proprietà fondamentali del calcolo classico delle probabilità, ne scende che tutti i risultati di tale calcolo non sono che *conseguenze* della definizione che abbiamo data della *coerenza*.

(Having proved the fundamental properties of the classical probability calculus, it follows that all its results are nothing else but *consequences* of the definition of *consistency* given in this paper.)¹

Indeed, a main consequence of de Finetti’s consistency theorem is that *the additivity axiom* for the disjunction of two incompatible events *becomes a consequence* of his consistency notion. As we will see, the consistency theorem is also the basis of the “product law” in the *definition* of independent events.

In Chapter 2, a proof of de Finetti’s consistency theorem is given. The mathematical background required is little more than what is taught in high school. As a preliminary step, in Chapter 1 we will prove what de Finetti called “the Fundamental Theorem of Probability”, his main contribution both to Boole’s probabilistic inference problem, and to its modern reformulation known as the optimization version of the probabilistic satisfiability problem PSAT. This is an immediate extension of the boolean satisfiability problem.

¹ The Italian adjective “coerente” (resp., the French “cohérent”) adopted by de Finetti in his original paper [2] (resp., in his paper [3]), is translated “consistent” here, for the sake of terminological consistency. As a matter of fact, the Italian translation of “logical consistency” is “coerenza logica”. Furthermore, de Finetti’s consistency is a generalization of logical consistency. See Corollary 2.3. Giandomenico Majone, the translator of most papers included in de Finetti’s book, “Probability, Induction and Statistics”, London: Wiley, 1972, uses “consistency” and not “coherence”.

Throughout these handouts we will use the terminology “consistency theorem” instead of “Dutch book theorem”. The latter denomination perhaps refers to non-British insurance companies that in past centuries used to make money “whatever the outcome of events”. Apart for its inelegance, this terminology has very little to do with de Finetti’s consistency theorem, where – if anything – it is the bettor who tries to have a positive gain whatever the outcome of events. Furthermore, as already noted, consistency as defined by de Finetti is a straightforward generalization of logical consistency. So our preference for the term “consistency” over “Dutch book” conforms to the standard mathematical practice of calling any deep characterization of a consistency notion a “consistency theorem”.

Chapter 3 is devoted to independence. There is a probability-free logical notion of independence, when we say that the existence of extra-terrestrial civilization is independent of the extinction of the black rhino over the next 30 years, without assigning numbers to these two events. Building on this notion of independence, we will show that the “product law” for the conjunction of two stochastically independent events is a consequence of de Finetti’s consistency theorem. Specifically, multiplication is the only operation that preserves the consistency of the betting odds posted by a bookmaker for conjunctions of events taken from two algebraically independent sets, such as results of football matches and horse racing.

In the final chapter we give a brief account of recent results on the *equivalence* between MV-algebras and their states, and Kolmogorov’s triples $(\Omega, \mathcal{B}(\Omega), \mu)$ for Ω a compact Hausdorff space and μ a regular probability measure on the space $\mathcal{B}(\Omega)$ of Borel sets of Ω .

This text is the result of a process of interaction between students of various cultural backgrounds and the author. He noticed that virtually all mathematical textbooks on probability omit – let alone prove – de Finetti’s consistency theorem. On the other hand, sifting through the mathematical articles devoted to proving this theorem, one can find proofs containing undefined notions, purported proofs of a “converse” of the theorem, (while de Finetti proved both directions), and mini-proofs resting on lemmas, mostly unknown in the early Thirties of the past century, for which the reader is referred to advanced texts on disparate topics.

Readers wishing to learn more about de Finetti’s contributions will draw from these handouts the necessary preparation to access the original proofs. These are written in the flowery style often found in his 1930s papers, which is different from the definition-theorem-proof style adopted here.

De Finetti’s enduring legacy will continue to inspire generations of probabilists, philosophers of science, logicians, statisticians and mathematical physicists. In these handouts we have presented the core concepts that students must understand in order to be able to make independent contributions.

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De Finetti's "Fundamental Theorem of Probability"

In his paper [2] de Finetti writes (in his own italics):

Un evento E è una proposizione, un'affermazione, che non sappiamo ancora se sia vera o falsa

An event E is a proposition, a statement, which we do not yet know whether it is true or false

De Finetti, [2, §7 p. 307]

Un individuo è coerente nel valutare le probabilità di certi eventi se qualunque gruppo di puntate $S_1, S_2, \dots, S_n$ un competitore faccia su un insieme qualunque di eventi $E_1, E_2, \dots, E_n$ fra quelli che egli ha considerato, non è possibile che il guadagno G del competitore risulti *in ogni caso positivo*.

An individual is consistent in evaluating the probabilities of certain events if for any set of stakes $S_1, S_2, \dots, S_n$ a competitor places on any set of events $E_1, E_2, \dots, E_n$ among those he has considered, it is not possible for the competitor's G gain to be *positive in any case*.

De Finetti, [2, §7 p. 308]

This expository style in presenting two of the most basic notions of de Finetti's theory differs from the style adopted in these handouts. The next few pages will be devoted to the *definition* of events and their outcomes in the context of boolean algebras, the *motivation* of these definitions, and the *representation* of events in euclidean finite-dimensional space. In Section 1.3 we will give concrete examples of events and their outcomes in our algebraic framework.

Events in Kolmogorov's axiomatic probability theory are members of boolean algebras closed under countably infinite disjunctions. In de Finetti's probability, events are closed only under finite disjunctions. In Chapter 4 we will see that these two interpretations of events are essentially equivalent.

1.1 Boolean algebras and their homomorphisms

While groups are a mathematical counterpart to “symmetries”, boolean algebras provide a rigorous approach to the imprecise notion of “event”. Furthermore, the homomorphisms of boolean algebras into the two-element boolean algebra $\{0, 1\}$ provide a convenient formal counterpart to the “possible outcomes” of these events.

Boolean algebras also provide an algebraic counterpart to “propositions” in boolean logic, equipped with the connectives of negation, conjunction and disjunction. Although logic is not the subject of these handouts, the reader will have various opportunities to see the mutual relationships between these two interpretations of boolean algebras.

An *algebra* is a nonempty set equipped with distinguished constants and operations.

Following standard practice, the mathematical neologism “iff” stands for “if and only if”. Thus for example, an even number is prime iff it equals 2. Sometimes we write $\Leftrightarrow$ instead of “iff”.

Definition 1.1. A *lattice* is an algebra $L = (L, \wedge, \vee)$ equipped with a partial order such that any two elements $x, y \in L$ have a greatest lower bound (alias the *infimum*, or *meet*) $x \wedge y$ and a least upper bound (also known as the *supremum*, or *join*) $x \vee y$. We say that L is *distributive* if for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

The *underlying order* of L is defined by the stipulation $x \leq y$ iff $x \wedge y = x$.

A *boolean algebra* $A = (A, 0, 1, \neg, \wedge, \vee)$ is a distributive lattice $(A, \wedge, \vee)$ with a smallest element 0 and a largest element 1, equipped with an operation $\neg$ (called *complementation*) satisfying $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.¹

An element $b \in A$ is said to be an *atom* if it is a nonzero minimal element, in symbols,

$$b \in \text{at}(A).$$

Example 1.2. The two-element boolean algebra $\{0, 1\} = (\{0, 1\}, \neg, \wedge, \vee)$ is defined by $\neg 0 = 1$, $\neg 1 = 0$, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$ for all $a, b \in \{0, 1\}$.

“Events” and their mutual relations will be understood as elements of a boolean algebra A acted upon by the operations of A .

¹ All boolean algebras in these handouts will satisfy the nontriviality condition $0 \neq 1$.

Definition 1.3. A *homomorphism* of a boolean algebra $A = (A, 0, 1, \neg, \wedge, \vee)$ into a boolean algebra $A' = (A', 0', 1', \neg', \wedge', \vee')$ is a function $\eta: A \rightarrow A'$ that preserves the complemented lattice structure:

$$\eta(\neg x) = \neg' x, \quad \eta(x \wedge y) = \eta(x) \wedge' \eta(y), \quad \eta(x \vee y) = \eta(x) \vee' \eta(y).$$

It follows that $\eta(0) = 0'$ and $\eta(1) = 1'$.

An *isomorphism* of A onto A' is a one-one function θ of A onto A' such that both θ and its inverse are homomorphisms.

Until further notice, $A = (A, 0, 1, \neg, \vee, \wedge)$ denotes a finite boolean algebra

This restriction will be removed in Section 2.4.

Elements a, b of a boolean algebra A are said to be *incompatible* if $a \wedge b = 0$.

Proposition 1.4. Let $x, y \in A$.

- (i) If $x \neq 0$ then x dominates some atom $b \in \text{at}(A)$. In symbols, $x \geq b$.
- (ii) For every $a \in \text{at}(A)$ exactly one of $a \leq x$ or $a \leq \neg x$ holds.
- (iii) $x \leq y$ iff $x \wedge \neg y = 0$.
- (iv) If $x < y$ then $y \wedge \neg x \neq 0$.
- (v) $\neg x$ is the largest element of A incompatible with x . Symmetrically, $\neg x$ is the smallest element $z \in A$ such that $x \vee z = 1$.
- (vi) For a function $\phi: A \rightarrow B$ to be an isomorphism of boolean algebras A and B , it is necessary and sufficient that ϕ is onto B and for all $x, y \in A$,

$$x \leq y \quad \text{iff} \quad \phi(x) \leq \phi(y). \quad (1.1)$$

Proof. (i) If x is an atom we are done. If x is not an atom, by definition of minimality there is a nonzero $x_1 < x$. If x_1 is an atom we are done. Otherwise there is nonzero $x_2 < x_1$. Since A is finite, any such chain $x > x_1 > x_2 > \dots$ will end after finitely many steps with a nonzero minimal element.

(ii) If (absurdum hypothesis) $a \leq x$ and $a \leq \neg x$ then by definition of infimum, $a \leq x \wedge \neg x = 0$ which contradicts $0 \neq a \in \text{at}(A)$. Thus *at most one* of the inequalities $a \leq x$ and $a \leq \neg x$ holds. To show that *at least one* inequality holds, let us assume $a \not\leq x$, i.e., $a \wedge x < a$, with the intent of proving $a \leq \neg x$. Since a is minimal nonzero,

$$a \wedge x = 0. \quad (1.2)$$

Furthermore,

$$a \wedge \neg x \neq 0. \quad (1.3)$$

For otherwise, combining $a \wedge \neg x = 0$ with condition (1.2) and the distributivity property of A , we obtain $0 = (a \wedge x) \vee (a \wedge \neg x) = a \wedge (x \vee \neg x) = a \wedge 1 = a$, which is impossible.

By (i) and (1.3) there is an atom $b \leq a \wedge \neg x$, i.e., $b \leq \neg x$ and $b \leq a$. It follows that $b = a$, whence $a \leq \neg x$, as desired.

(iii) ($\Rightarrow$) From the assumption $x \wedge y = x$ we have $x \wedge \neg y = x \wedge y \wedge \neg y = 0$. ($\Leftarrow$) We first write $y \vee (\neg y \wedge x) = (y \vee \neg y) \wedge (y \vee x) = 1 \wedge (y \vee x) = y \vee x$. From our standing hypothesis $x \wedge \neg y = 0$ we obtain $y \vee x = y \vee (\neg y \wedge x) = y \vee 0 = y$ whence $y \geq x$.

(iv) By (iii), $x \neq y$ iff either $x \wedge \neg y \neq 0$ or $y \wedge \neg x \neq 0$. Therefore, $x < y$ iff $y \neq x$ and $x \leq y$ iff (either $x \wedge \neg y \neq 0$ or $y \wedge \neg x \neq 0$) and $x \wedge \neg y = 0$ iff $y \wedge \neg x \neq 0$ and $x \wedge \neg y = 0$, which implies $y \wedge \neg x \neq 0$.

(v) Distributivity ensures that the supremum s of (the finite set of) all elements of A incompatible with x is also incompatible with x . Therefore, s is the greatest element of A incompatible with x . In particular, since $\neg x$ is incompatible with x , $s \geq \neg x$. By way of contradiction, assume $s > \neg x$. Then from (iv) we can write $0 \neq s \wedge \neg \neg x = s \wedge x = 0$, which is impossible. So $s = \neg x$.

The rest is proved similarly.

(vi) If ϕ is an isomorphism then it is onto B , and both ϕ and its inverse preserve the lattice-order structure, whence (1.1) is satisfied. Conversely, assume $\phi: A \rightarrow B$ is onto B and satisfies (1.1). It follows that ϕ is one-one. Further, both ϕ and ϕ^{-1} preserve the lattice operations. By the characterization of $\neg x$ in (v), ϕ and ϕ^{-1} also preserve the $\neg$ operation. $\square$

The symbol $\square$ stands for the end of a proof.

Representation of finite boolean algebras as powerset algebras

Let $\emptyset$ be shorthand for the empty set. For any boolean algebra A let

$$\text{pow}(\text{at}(A)) \quad (\text{read: "the powerset of } \text{at}(A)\text{"}) \quad (1.4)$$

be the boolean algebra of all subsets of $\text{at}(A)$, where $0 = \emptyset$, $1 = \text{at}(A)$ and for all subsets X, Y of $\text{at}(A)$,

$$X \vee Y = X \cup Y, \quad X \wedge Y = X \cap Y, \quad \neg X = \text{at}(A) \setminus X = \text{the complement of } X.$$

Proposition 1.5. *For any $x \in A$ let $\downarrow x$ be the set of atoms dominated by x ,*

$$\downarrow x = \{a \in \text{at}(A) \mid a \leq x\}.$$

We then have

(i) $\downarrow y = \emptyset$ iff $y = 0$. If $\downarrow x = \{b_1, \dots, b_l\} \neq \emptyset$ then $x = b_1 \vee \dots \vee b_l$.

(ii) The function $\downarrow$ is an isomorphism of A onto $\text{pow}(\text{at}(A))$.

Proof. By Proposition 1.4(i), the first statement is trivial. For the second statement, by definition of supremum, $x \geq \bigvee_{i=1}^l b_i$. By way of contradiction, assume $x > \bigvee_{i=1}^l b_i$. By Proposition 1.4(iv), $x \wedge \neg \bigvee_{i=1}^l b_i \neq 0$, whence by Proposition 1.4(i) some atom b is dominated by both x and $\neg \bigvee_{i=1}^l b_i$. Since $\{b_1, \dots, b_l\}$ is the list of all atoms dominated by x , we may safely assume $b = b_1$. Then $b_1 \leq \neg \bigvee_{i=1}^l b_i = \bigwedge_{i=1}^l \neg b_i$, whence $b_1 \leq \neg b_1$. Thus $b_1 \leq b_1 \wedge \neg b_1 = 0$, which is impossible.

(ii) From (i) we have

$$\text{for any } \{b_1, \dots, b_l\} \subseteq \text{at}(A), \quad \downarrow (b_1 \vee \dots \vee b_l) = \{b_1, \dots, b_l\}. \quad (1.5)$$

Thus the function $\downarrow$ is onto $\text{pow}(\text{at}(A))$. In the light of Proposition 1.4(vi), there remains to be proved that for all $x, y \in A$, $x \leq y$ iff $\downarrow x \subseteq \downarrow y$. If $x \leq y$ then every atom dominated by x is also dominated by y , and hence, $\downarrow x \subseteq \downarrow y$. Conversely, if $x \not\leq y$ then $x \wedge \neg y \neq 0$ by Proposition 1.4(iii). Proposition 1.4(i) yields an atom a such that $a \leq x \wedge \neg y$. Since $a \leq \neg y$, then by Proposition 1.4(ii), $a \not\leq y$, whence $a \notin \downarrow y$. Since $a \leq x$, $a \in \downarrow x$. In conclusion, $\downarrow x \not\subseteq \downarrow y$. $\square$

Part (ii) in Proposition 1.5 is an example of a “representation theorem”. Its role is to give the reader a concrete realization of every finite boolean algebra A as the boolean algebra of all subsets of the set of atoms of A , equipped with union, intersection and complement.

1.2 The geometry of finite boolean algebras in $\mathbb{R}^n$

As usual, $\mathbb{R}$ denotes the real line, $\mathbb{R}^2$ the cartesian plane, and $\mathbb{R}^n$ the n -dimensional euclidean space. We let $[0, 1]^n$ denote the *unit n -cube* in $\mathbb{R}^n$. Then $\{0, 1\}^n$ is the set of vertices of $[0, 1]^n$. The *standard basis vectors* of $\mathbb{R}^n$ are denoted $e_1, \dots, e_n$. Throughout these handouts, such basic facts as the linear independence of the basis vectors of $\mathbb{R}^n$ will be used tacitly.

Definition 1.6. The *product order* of $\{0, 1\}^n$ is defined by stipulating that $x \leq y$ if and only if, whenever a coordinate x_i of x is 1 then so is the coordinate y_i of y .

Geometrically, $x < y$ means that there is a path from x to y consisting of steps along consecutive edges of the unit cube $[0, 1]^n$, where each step moves away from the origin.

A direct inspection yields:

Proposition 1.7. *The product order makes $\{0, 1\}^n$ into a boolean algebra with bottom element $0 = \text{the origin in } \mathbb{R}^n$ and top element $1 = (1, \dots, 1)$. For any $x, y \in \{0, 1\}^n$ the i th coordinate $(\neg x)_i$ of $\neg x$ equals $1 - x_i$. Further, $(x \wedge y)_i = \min(x_i, y_i)$ and $(x \vee y)_i = \max(x_i, y_i)$. The atoms of $\{0, 1\}^n$ are the standard basis vectors $e_1, \dots, e_n$, $\text{at}(\{0, 1\}^n) = \{e_1, \dots, e_n\}$.*

Theorem 1.8. (Geometric representation of finite boolean algebras in $\mathbb{R}^n$)
 With $\{a_1, \dots, a_n\} = \text{at}(A)$ and $\{e_1, \dots, e_n\}$ the standard basis vectors in $\mathbb{R}^n$,
 let the function $\iota: A \rightarrow \{0, 1\}^n$ be defined by stipulating that for all $x \in A$,

$$\iota(x) = \bigvee \{e_i \mid a_i \leq x\} = \sum \{e_i \mid a_i \leq x\} \in \mathbb{R}^n. \quad (1.6)$$

Then ι is an isomorphism of A onto $\{0, 1\}^n$, $\iota: A \cong \{0, 1\}^n$.

Proof. By (1.5) and Proposition 1.5, the linear independence of the standard basis vectors ensures that ι is one-one. To prove that every $v \in \{0, 1\}^n$ is in the range of ι , let us write

$$v = e_{n_1} + \dots + e_{n_k} \quad (1.7)$$

for a unique subset $\{e_{n_1}, \dots, e_{n_k}\}$ of $\{e_1, \dots, e_n\}$. Let $\{a_{n_1}, \dots, a_{n_k}\}$ be the corresponding set of atoms of A , and $x_v = a_{n_1} \vee \dots \vee a_{n_k}$. By Proposition 1.5(i), $\downarrow x_v = \{a_{n_1}, \dots, a_{n_k}\}$. By (1.6), $\iota(x_v) = v$, which shows that the function ι is onto $\{0, 1\}^n$.

In view of Proposition 1.4(vi), there remains to be proved that for all $x, y \in A$ $x \leq y \Leftrightarrow \iota(x) \leq \iota(y)$. By Definition 1.6 and (1.6), for all $i = 1, \dots, n$ we have

$$e_i \leq \iota(x) \Leftrightarrow a_i \leq x.$$

As a consequence,

$$x \leq y \Leftrightarrow \downarrow x \subseteq \downarrow y, \text{ by Proposition 1.5(ii)}$$

$$\Leftrightarrow \text{for all } a \in \text{at}(A) \text{ such that } a \leq x \text{ we have } a \leq y$$

$$\Leftrightarrow \text{for all } e \in \text{at}(\{0, 1\}^n) \text{ such that } e \leq \iota(x) \text{ we have } e \leq \iota(y)$$

$$\Leftrightarrow \iota(x) \leq \iota(y). \quad \square$$

Geometric representation of the homomorphisms of A into $\{0, 1\}$

For every (always column) vector $v \in \mathbb{R}^n$ we let v^T denote its transpose. Thus, e.g., if v is the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ then $v^T = (x, y)$. For any vector w in $\mathbb{R}^n$ we let $v^T w$ denote the matrix multiplication of v^T and w , i.e., their scalar product in $\mathbb{R}^n$.

For any two (possibly infinite) sets X and Y , by a *one-one correspondence* between X and Y we mean an injective function $f: X \rightarrow Y$ with $f(X) = Y$.

We use the notation

$$\text{hom}(A)$$

for the set of homomorphisms of A into the two-element boolean algebra $\{0, 1\}$. It follows that any $\eta \in \text{hom}(A)$ satisfies the identities

$$\eta(\neg x) = 1 - \eta(x), \quad \eta(x \wedge y) = \min(\eta(x), \eta(y)), \quad \eta(x \vee y) = \max(\eta(x), \eta(y)).$$

Intuitively, the homomorphisms of A into $\{0, 1\} = \{no, yes\}$ are the “possible outcomes” of the “events” (i.e., the elements) of A .

The most general example of a homomorphism of A into $\{0, 1\}$ is given by the following result, which in general fails when A is an infinite boolean algebra:

Corollary 1.9. (Geometric representation of $\text{hom}(A)$) *For any A we have:*

(i) *Let the function $a \mapsto \eta_a$ send every atom a into the function $\eta_a: A \rightarrow \{0, 1\}$ such that for all $x \in A$, $\eta_a(x) = 1$ iff $a \leq x$. Then $a \mapsto \eta_a$ is a one-one correspondence between $\text{at}(A)$ and $\text{hom}(A)$. The inverse correspondence sends any $\eta \in \text{hom}(A)$ to the only atom a_η of A such that $\eta(a_\eta) = 1$.*

(ii) *Let $\{e_1, \dots, e_n\} = \text{at}(\{0, 1\}^n)$. For each $i = 1, \dots, n$ let the function $\theta_{e_i}: \{0, 1\}^n \rightarrow \{0, 1\}$ be defined by stipulating that for every $v \in \{0, 1\}^n$*

$$\theta_{e_i}(v) = 1 \text{ iff } e_i \leq v \text{ in the product order of } \{0, 1\}^n. \quad (1.8)$$

Then the function $e_i \mapsto \theta_{e_i}$, ($i = 1, \dots, n$) is a one-one correspondence between $\text{at}(\{0, 1\}^n)$ and $\text{hom}(\{0, 1\}^n)$. For any atom e of $\{0, 1\}^n$, θ_e is the only homomorphism of $\text{hom}(\{0, 1\}^n)$ such that $\theta_e(e) = 1$. The inverse correspondence sends every $\theta \in \text{hom}(\{0, 1\}^n)$ to the uniquely determined standard basis vector $e_\theta \in \text{at}(\{0, 1\}^n) \subseteq \mathbb{R}^n$ such that $\theta(e_\theta) = 1$.

(iii) *For every $v \in \{0, 1\}^n$ and $i = 1, \dots, n$,*

$$\theta_{e_i}(v) = e_i^T v = i\text{th coordinate of } v. \quad (1.9)$$

Proof. (i) For all $x, y \in A$, $\eta_a(x \wedge y) = 1$ iff $x \wedge y \geq a$ iff $x \geq a$ and $y \geq a$ iff $\eta_a(x) = 1$ and $\eta_a(y) = 1$. Thus $\eta_a(x \wedge y) = \eta_a(x) \wedge \eta_a(y)$. Similarly, η_a preserves the $\vee$ operation. Preservation of the $\neg$ operation follows from Proposition 1.4(ii). Therefore, $\eta_a \in \text{hom}(A)$. If a and b are distinct atoms of A then $\eta_a(a) = 1$ and $\eta_b(a) = 0$, because $a \not\leq b$. Thus the function $a \mapsto \eta_a$ is one-one. For any $\eta \in \text{hom}(A)$ let $B_\eta \subseteq \text{at}(A)$ be the set of atoms b such that $\eta(b) = 1$. B_η is nonempty, for otherwise (absurdum hypothesis),

$$1 = \eta(1) = \eta(\bigvee_{i=1}^n a_i) = \max\{\eta(a) \mid a \in \text{at}(A)\} = 0,$$

which is impossible. We have just proved that the function $a \mapsto \eta_a$ is onto $\text{hom}(A)$. Finally, B_η cannot contain two distinct atoms a, b . For otherwise, (absurdum hypothesis), Proposition 1.4(ii) yields $b \leq \neg a$, whence $0 = 1 - \eta(a) = \eta(\neg a) \geq \eta(b) = 1$, a contradiction. So precisely one atom a_η belongs to B_η . Evidently, $\eta = \eta_{a_\eta}$.

(ii) By Theorem 1.8, this is the special case of (i) for $A = \{0, 1\}^n$.

(iii) For arbitrary $v \in \{0, 1\}^n$, let $e_{n_1}, \dots, e_{n_k} \in \mathbb{R}^n$ be the basis vectors $\leq v$. By Proposition 1.5(i) and (1.7) we have $v = e_{n_1} + \dots + e_{n_k}$. From (1.8) we obtain

$$\begin{aligned}
\theta_{e_i}(v) = 1 & \text{ iff } e_i \leq v, \quad (1.8) \\
& \text{ iff } e_i \in \{e_{n_1}, \dots, e_{n_k}\}, \text{ by Definition 1.6} \\
& \text{ iff } e_i^T(e_{n_1} + \dots + e_{n_k}) = 1 \\
& \text{ iff } e_i^T v = 1.
\end{aligned}$$

□

1.3 Events, atomic events, and possible worlds

Fix an integer $n = 1, 2, \dots$ and a set $G = \{X_1, \dots, X_n\}$. Does there exist a largest boolean algebra containing G as a generating set?

Take a coin, toss it n times, and record the result (head=1 or tail=0) of each toss. Suppose for each $i = 1, \dots, n$, X_i stands for the event "the result of the i th toss of my coin is head". With $\neg, \wedge, \vee$ the usual connectives of boolean logic, let us consider the 2^n boolean formulas (called *miniterms*)

$$X_1^{\beta_1} \wedge \dots \wedge X_n^{\beta_n} \text{ with } X_i^{\beta_i} = X_i \text{ if } \beta_i = 1, \text{ and } X_i^{\beta_i} = \neg X_i \text{ if } \beta_i = 0. \quad (1.10)$$

These miniterms record any possible outcome of your n tosses of a coin. Each miniterm stands for a sequence of n "independent" events, in the sense that the occurrence or non-occurrence of X_i does not interfere with the occurrence or non-occurrence of X_j , ($i \neq j$).

Let F_n be the set of all boolean *formulas* in the *variables* $X_1, \dots, X_n$, where two formulas are identified iff they are logically equivalent: thus for instance, $\neg\neg X_1$ is identified with X_1 , $X_1 \wedge X_2$ is identified with $X_2 \wedge X_1$, and $X_1 \vee (X_2 \wedge X_3)$ is identified with $(X_1 \vee X_2) \wedge (X_1 \vee X_3)$.²

The result is the *free boolean algebra on the free generating set* $G = \{X_1, \dots, X_n\}$.

Readers who (like de Finetti) have little propensity for logic may adopt any of the following two alternative definitions of F_n :

$$G \text{ generates } F_n \text{ and } X_1^{\beta_1} \wedge \dots \wedge X_n^{\beta_n} \neq 0 \text{ for all } (\beta_1, \dots, \beta_n) \in \{0, 1\}^n;$$

or, equivalently,

$$G \text{ generates } F_n, \text{ and for every boolean algebra } A \text{ and function } f: G \rightarrow A, f \text{ uniquely extends to a homomorphism of } F_n \text{ into } A.$$

As expected, F_n is the largest possible boolean algebra containing G as a generating set. For, if F is another boolean algebra generated by G , then by our last definition of F_n , the identity function $\epsilon: X_i \mapsto X_i$ extends to a homomorphism $\tilde{\epsilon}$ of F_n into F . Now, $\tilde{\epsilon}$ is onto F , because the X_i generate F . We conclude that $|F| \leq |F_n|$.

² By assigning 0 or 1 to the variables in all possible ways and working in the two-element boolean algebra $\{0, 1\}$ one has a familiar mechanical procedure to check if two formulas are logically equivalent.

The miniterms of F_n in (1.10) are the 2^n atoms of F_n . Intuitively, they are the “atomic events” of F_n . As in Corollary 1.9, each miniterm $t \in F_n$ uniquely determines a “possible world” of F_n , i.e., a homomorphism F_n of into $\{0, 1\}$, assigning 1 or 0 to any event $e \in F_n$, according as e dominates the atom t or is disjoint from t . In Proposition 1.4(ii) it is shown that this alternative always occurs. Conversely, any $\eta \in \text{hom}(F_n)$ uniquely determines the atom a_η given by the smallest element $a \in F_n$ such that $\eta(a) = 1$. The independence of the events X_i results in the largest set of possible worlds.³

Now suppose $n = 3$ and X_1, X_2, X_3 stand for the following events, where “wins” is shorthand for “wins the next FIFA Club World Cup”:

$$X_1 = \text{Brazil wins}, \quad X_2 = \text{Spain wins}, \quad X_3 = \text{France wins}.$$

From the rules of the FIFA World Cup it follows that not all 2^3 atomic events of F_3 , coded by the miniterms (1.10), can occur. For instance, the atomic event $X_1 \wedge X_2 \wedge \neg X_3$ is impossible, and so is, a fortiori, the atomic event $X_1 \wedge X_2 \wedge X_3$. The impossible atomic events are precisely those stating that the number of winners is ≥ 2 . On the other hand, $\neg X_1 \wedge \neg X_2 \wedge \neg X_3$ is possible. It follows that the boolean algebra A generated by the three events X_1, X_2, X_3 is strictly smaller than the free three-generator algebra F_3 , the largest possible boolean three-generator algebra. We construct A by deleting from the set of atoms of F_3 those which code atomic events forbidden by the rules of the FIFA cup. A moment’s reflection shows that the surviving four atomic events in A are as follows:

$$X_1 \wedge \neg X_2 \wedge \neg X_3, \quad \neg X_1 \wedge X_2 \wedge \neg X_3, \quad \neg X_1 \wedge \neg X_2 \wedge X_3, \quad \neg X_1 \wedge \neg X_2 \wedge \neg X_3.$$

As expected, the boolean algebra A has $2^4 = 16$ elements/events, fewer than the $2^8 = 256$ elements of F_3 . Beyond X_1, X_2, X_3 themselves, and the four atoms of A , examples of events of A include $\neg X_1 \wedge X_2$ (which in A is the same as X_2), the impossible event $X_1 \wedge X_2$ (i.e., the zero element of A), the sure event $X_1 \vee \neg X_1$ (i.e., the top element 1 of A), and a few others.

As in Proposition 1.5(i), each event of A is the disjunction (algebraically speaking, the supremum) of the atomic events it dominates. Thus, e.g., in A we have

$$\neg X_1 = (\neg X_1 \wedge \neg X_2 \wedge X_3) \vee (\neg X_1 \wedge X_2 \wedge \neg X_3) \vee (\neg X_1 \wedge \neg X_2 \wedge \neg X_3),$$

while an easy exercise shows that in F_3 the element $\neg X_1$ is the supremum of four atoms.

³ Kolmogorov calls each atom of F_n an “elementary event”. For Boole, the atoms of F_n are its “constituents”. De Finetti says that each atomic event is a “case”. By Stone’s theorem 3.4, the 2^n atoms of F_n correspond to the points (also known as “sample points”) of the dual space of F_n . As we will see, an infinite boolean algebra A need not have atoms. The set $\text{hom}(A)$ conveniently replaces the set of atoms of A in any case.

By Proposition 1.5(ii), up to isomorphism, every n -generator boolean algebra arises from a similar reduction procedure of the atoms of a free boolean algebra F_n .

1.4 States

Definition 1.10. A *state* of a boolean algebra A is a function $\sigma: A \rightarrow [0, 1]$ with $\sigma(1) = 1$, having the *additivity property*: For all $x, y \in A$ if $x \wedge y = 0$ then $\sigma(x \vee y) = \sigma(x) + \sigma(y)$.

An easy verification shows that *every homomorphism of A into $\{0, 1\}$ is a state*.

The geometric representation of every state σ of a boolean algebra A is the object of the following corollary, where σ is identified with a suitable vector in euclidean space, and the value assigned by σ to an event is the scalar product of the vector representing the state and the vector representing the event.

A (possibly infinite) set X in euclidean space $\mathbb{R}^n$ is *convex* if for any two points $x, y \in X$ the segment joining x and y is contained in X .

For any set $\{x_1, \dots, x_l\} \subseteq \mathbb{R}^n$, the *convex hull* $\text{conv}(x_1, \dots, x_l)$ is the set of all *convex combinations* of $x_1, \dots, x_l$, i.e., all points x of $\mathbb{R}^n$ of the form

$$x = \lambda_1 x_1 + \dots + \lambda_n x_n, \quad 0 \leq \lambda_i \in \mathbb{R}, \quad (i = 1, \dots, n), \quad \sum_{i=1}^n \lambda_i = 1.$$

An element $z \in X$ is *extremal* (in X) if whenever $z \in \text{conv}(a, b)$ for some $a, b \in X$ then $z = a$ or $z = b$.

Corollary 1.11. (Geometric representation of the states of A) *Let σ be a state of the boolean algebra $\{0, 1\}^n$. Let the closed convex set $S \subseteq \mathbb{R}^n$ and the vector $v_\sigma \in \mathbb{R}^n$ be respectively defined by*

$$S = \text{conv}(e_1, \dots, e_n) \quad \text{and} \quad v_\sigma = (\sigma(e_1), \dots, \sigma(e_n)). \quad (1.11)$$

Then

- (i) $(v_\sigma)_1 + \dots + (v_\sigma)_n = 1$. Thus $v_\sigma \in S$.
- (ii) For all $v = (v_1, \dots, v_n) \in \{0, 1\}^n$, $\sigma(v) = v_\sigma^T v$.
- (iii) v_σ is the only vector in $\mathbb{R}^n$ such that for all $v \in \{0, 1\}^n$, $\sigma(v) = v_\sigma^T v$.
Therefore, any state is uniquely determined by the values it assigns to the atoms e_i .

Proof. (i) The top element 1 in the boolean algebra $\{0, 1\}^n$ is the sum $e_1 + \dots + e_n = e_1 \vee \dots \vee e_n$. Since $e_i \wedge e_j = 0$ for $i \neq j$ then 1 is the supremum of incompatible elements of $\{0, 1\}^n$. By additivity, $1 = \sigma(\bigvee_{i=1}^n e_i) = \sum_{i=1}^n \sigma(e_i)$, whence $v_\sigma \in S$.

(ii) For each $i = 1, \dots, n$ let us agree to write $0e_i =$ the origin in $\mathbb{R}^n$ and $1e_i = e_i$. The vector v is a linear combination of the pairwise orthogonal vectors $e_1, \dots, e_n$ with uniquely determined coefficients 1 or 0 according as $e_i \leq v$ or $e_i \not\leq v$. Further, v is the supremum of the e_i with the same coefficients. By Corollary 1.9(iii), $v_i = \theta_{e_i}(v) \in \{0, 1\}$, whence

$$v = \sum_{i=1}^n \theta_{e_i}(v) e_i = \bigvee_{i=1}^n \theta_{e_i}(v) e_i.$$

By (1.11) and the additivity property of σ we can write

$$\begin{aligned} \sigma(v) &= \sigma\left(\bigvee_{i=1}^n \theta_{e_i}(v) e_i\right) = \sum_{i=1}^n \sigma(\theta_{e_i}(v) e_i) = \sum_{i=1}^n \theta_{e_i}(v) \sigma(e_i) \\ &= \sum_{i=1}^n v_i (v_\sigma)_i = v^T v_\sigma = v_\sigma^T v. \end{aligned}$$

(iii) Suppose $w \in \mathbb{R}^n$ satisfies $w^T v = \sigma(v)$ for all $v \in \{0, 1\}^n$. Then by (i)-(ii), for all $i = 1, \dots, n$, we have $(w^T - v_\sigma^T) e_i = 0$ whence $w_i = (v_\sigma)_i$ and $w = v_\sigma$. $\square$

1.5 De Finetti's "Fundamental Theorem of Probability"

In his "Investigation of the laws of thought", [1, Chapter XVI, 4, p. 246], Boole writes:

"the object of the theory of probabilities might be thus defined. Given the probabilities of any events, of whatever kind, to find the probability of some other event connected with them."

In his paper [2] de Finetti gave a criterion for the probability of the new event to exist. This is his consistency theorem 2.2, based on Definition 2.1. Furthermore, in [3, p. 13] and on page 112 of his book [4, 3.10.1], de Finetti showed that the set of possible probabilities of the new event (if nonempty) is a closed interval contained in $[0, 1]$.

He named his result "the Fundamental Theorem of Probability".

For our self-contained proof of this theorem we prepare the following notation and terminology, which will also be used in a later chapter: A (closed) *hyperplane* H in euclidean space $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $\{x \in \mathbb{R}^n \mid$

$l^T x = w\}$ for some nonzero vector $l \in \mathbb{R}^n$ and real number w . The sets $\{x \in \mathbb{R}^n \mid l^T x \geq w\}$ and $\{x \in \mathbb{R}^n \mid l^T x \leq w\}$ are the two (closed) half-spaces with boundary H .

Theorem 1.12. (De Finetti's "Fundamental Theorem of Probability") *Let A be a boolean algebra with n atoms and state space $S(A)$. Fix a finite subset $E = \{h_1, \dots, h_m\}$ of A along with a function $\beta: E \rightarrow [0, 1]$, and let*

$$S_\beta = \{\sigma \in S(A) \mid \sigma \supseteq \beta\} \text{ be the set of states of } A \text{ extending } \beta.$$

Next, for any $h \in A$ let

$$S_\beta(h) = \{\sigma(h) \mid \sigma \in S_\beta\} \quad (1.12)$$

be the set of possible values assigned to h by all states extending β . Then $S_\beta(h)$ is either empty, or is a closed interval contained in $[0, 1]$ possibly consisting of a single point.

Proof. In view of the representation Theorem 1.8, we can identify A with the boolean algebra $\{0, 1\}^n$. Then E is a set of vertices $h_j \in \{0, 1\}^n$ of the n -cube $[0, 1]^n$. For all $\sigma \in S(A)$ and $w \in \{0, 1\}^n$, the value $\sigma(w)$ is given by the scalar product $v_\sigma^T w$, with $v_\sigma = (\sigma(e_1), \dots, \sigma(e_n))$ the vector associated to σ by the geometric representation in Corollary 1.11. In particular, v_σ lies in the closed convex set

$$S = \text{conv}(e_1, \dots, e_n), \text{ with } e_1, \dots, e_n \text{ the standard basis vectors of } \mathbb{R}^n.$$

For every $j = 1, \dots, m$, the set $\{\sigma \in S(A) \mid \sigma(h_j) = \beta(h_j)\}$ of states of A extending the singleton function $h_j \mapsto \beta(h_j)$ is the set of vectors

$$\{v_\sigma \in S \mid \sigma(h_j) = \beta(h_j)\} = \{v_\sigma \in S \mid v_\sigma^T h_j = \beta(h_j)\}.$$

This is the intersection of S with the hyperplane $H_j \subseteq \mathbb{R}^n$ of all vectors $w \in \mathbb{R}^n$ such that $w^T h_j = \beta(h_j)$. As a consequence, the set $S_\beta \subseteq S$ of states of A extending β coincides with $S \cap H_1 \cap \dots \cap H_m$. It follows that S_β is a closed convex (a fortiori connected) subset of S . The set $S_\beta(h)$ in (1.12) has the form $\{v_\sigma^T h \in [0, 1] \mid \beta \subseteq \sigma \in S\}$. Thus $S_\beta(h)$ is the range of the continuous $[0, 1]$ -valued function $v_\sigma \mapsto v_\sigma^T h$, where the vector v_σ ranges over the closed connected space S_β . Elementary topology shows that $S_\beta(h)$ is closed and connected. In conclusion, if nonempty, $S_\beta(h)$ is a closed interval in $[0, 1]$. $\square$

Remarks 1.13 Boole's remarks on the object of the theory of probabilities are taken up today in the PSAT (probabilistic satisfiability) problem and its optimization version. Here events $h_1, \dots, h_m$ are assigned probabilities $p_1, \dots, p_m$, and one has to specify the possible values of the probability p_{m+1} of a new event h_{m+1} . For computational purposes, all events p_i are coded by *boolean formulas*, and all $p_1, \dots, p_m$, are *rational* numbers.

Theorem 1.12 shows that the set of possible values of p_{m+1} , if nonempty, is a closed interval contained in $[0, 1]$. To provide a fundamental necessary and sufficient condition for this set to be nonempty, it will take de Finetti's consistency ($\sim$ Dutch book) theorem 2.2. PSAT is the problem of checking if this condition is valid. PSAT is a generalization of the satisfiability problem SAT for boolean formulas, and has the same computational complexity as SAT.

PSAT and its optimization version pertain to a vibrant research area in various domains, including defeasible reasoning, automated deduction, formal epistemology, and uncertainty management. For more information, also including de Finetti's contributions to Boole's problem and its modern reformulation, see P. Hansen, B. Jaumard, "Probabilistic Satisfiability", in: Handbook of Defeasible Reasoning and Uncertainty Management Systems, Vol. 5. (J. Kohlas, S. Moral, Editors). Springer, 2000, pp. 321-367.

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De Finetti's Consistency Theorem

Lot T. Tery is an honest and experienced manager of a worldwide lottery. Each ticket reads:

I, the undersigned seller of this ticket, will buy it back paying the bearer one euro, if Spain wins the next FIFA world cup.
(Signed: the seller)

The price of each ticket is p (euro, to fix ideas).

If p is a fair price for you and you decide to buy N tickets, you now pay Np to Lot T. Tery. He, the signatory ticket seller, will pay you N if Spain wins.

On the other hand, if the ticket price seems too high to you, why not ask Lot T. Tery to *buy* M tickets from you? What better proof that the price p is right? His acclaimed honesty makes him willing to exchange his managerial role with you: he pays you pM for M tickets. You, the signatory seller of the ticket, will pay him M if Spain wins.

As de Finetti notes on page 309 of his paper [1], the price p set by any experienced manager like Lot T. Tery must satisfy the trivial inequalities $0 \leq p \leq 1$. Otherwise, you could bankrupt him, whatever the outcome of the next FIFA world cup. As a matter of fact, if $p > 1$, selling him Z tickets, with $Z = \text{one zillion}$, you will get $(p - 1)Z$ if Spain wins and pZ otherwise. Lot T. Tery will get even worse if he sells Z tickets at a price $p < 0$.

Therefore, first of all, consistency requires that the ticket price be in the range $[0, 1]$. Any inconsistency on Lot T. Tery's part can be punished by you, sending him to ruin, whatever the outcome of the FIFA cup.

What other conditions must the ticket price p meet, to keep you from bankrupting Lot T. Tery? Definition 2.1 provides the answer, even for the general case where tickets for multiple events and sold/bought. De Finetti's consistency theorem 2.2 gives a characterization of consistency.

2.1 Consistency

Following de Finetti, we define the fundamental notion of a consistent assignment of numbers in $[0, 1]$ to “events” $h_1, \dots, h_m$ understood as elements of a finite boolean algebra A . The finiteness hypothesis will be dropped in Section 2.4.

With $\{a_1, \dots, a_n\}$ the set of atoms of A , let $\{\eta_{a_1}, \dots, \eta_{a_n}\} = \{\eta_1, \dots, \eta_n\} = \text{hom}(A)$ be the set of homomorphisms of A into the two-element boolean algebra $\{0, 1\}$, as in Corollary 1.9(i).

Definition 2.1. (De Finetti, [1, pp. 304-305]) Let $E = \{h_1, \dots, h_m\}$ be a subset of A and $\beta: E \rightarrow [0, 1]$ a function. Then β is said to be *inconsistent in* A if there exists a function $s: E \rightarrow \mathbb{R}$ such that

$$\sum_{j=1}^m s(h_j)(\beta(h_j) - \eta_i(h_j)) < 0 \quad \text{for every } i = 1, \dots, n. \quad (2.1)$$

If β is not inconsistent in A we say it is *consistent in* A .

The rationale of Definition 2.1, [1, §4]: Bookie vs. Betty

To better understand Definition 2.1 and its characterization in theorem 2.2, recalling de Finetti's second citation at the outset of Chapter 1, let us replace Lot T. Tery by a bookmaker named Bookie grappling with a clever bettor named Betty. Bookie posts “betting odds”, or “betting rates”

$$p_1 = \beta(h_1), \dots, p_n = \beta(h_m) \in [0, 1]$$

on future events $h_1, \dots, h_m \in A$. As in the example above, $p_j = \beta(h_j)$ is the price of each ticket for a payoff of 1 euro in case h_i occurs, and 0 otherwise.¹

If Bookie's price $\beta(h_j)$ is deemed reasonable by Betty, she now places a stake $s(h_j) \geq 0$, pays $\beta(h_j)s(h_j)$, hoping to win $s(h_j)$ if h_j occurs. It goes without saying that if h_j does not occur Betty's win will be zero.

On the other hand, if Betty finds $\beta(h_j)$ excessive — as is almost always the case with bookmakers' odds in real life — *she can place a stake* $s(h_j) < 0$. So Bookie pays Betty $\beta(h_j)|s(h_j)|$, and wins $|s(h_j)|$ if h_j occurs. In this case, Bookie/Betty's roles swap. This never happens in real life.

For definiteness, let us make the following stipulation:

$$\text{All financial transactions are given the Betty-to-Bookie orientation.} \quad (2.2)$$

Then for any positive or negative stake $s(h_j)$, the balance of this single bet on event h_j is given by $s(h_j)(\beta(h_j) - \eta_i(h_j))$ in the “possible world” $\eta_i \in$

¹ Real-life bookmakers prefer to post their odds as $1/p_j \geq 1$ instead of p_j (always > 0), guess why.

$\text{hom}(A)$.² Since zero bets are possible, we may assume Betty is betting on all events $h_1, \dots, h_m$. Bookie's balance is then

$$\sum_{j=1}^m s(h_j)(\beta(h_j) - \eta_i(h_j)). \quad (2.3)$$

Bookie's book β is inconsistent in the sense of Definition 2.1 iff Betty can devise positive or negative stakes $s(h_1), \dots, s(h_m) \in \mathbb{R}$ that guarantee her a minimum profit of 1 — equivalently, one zillion — “regardless of the outcome of the events h_j ”, i.e., in any “possible world” $\eta_i \in \text{hom}(A)$. Correspondingly, Bookie has a sure loss of at least one(zillion).

The totality of “possible outcomes”, or “possible worlds”, is made precise by the homomorphisms $\eta_1, \dots, \eta_n \in \text{hom}(A)$. Definition 1.3 ensures that the laws of logic hold in each η_i .

To view panoramically all events h_j , ($j = 1, \dots, m$) and all possible worlds η_i , ($i = 1, \dots, n$), we prepare the following $n \times m$ matrix M :

$$M = \begin{pmatrix} \beta(h_1) - \eta_1(h_1) & \beta(h_2) - \eta_1(h_2) & \dots & \beta(h_m) - \eta_1(h_m) \\ \beta(h_1) - \eta_2(h_1) & \beta(h_2) - \eta_2(h_2) & \dots & \beta(h_m) - \eta_2(h_m) \\ \dots & \dots & \dots & \dots \\ \beta(h_1) - \eta_n(h_1) & \beta(h_2) - \eta_n(h_2) & \dots & \beta(h_m) - \eta_n(h_m) \end{pmatrix} \quad (2.4)$$

Let $s(h_1), \dots, s(h_m) \in \mathbb{R}$ be Betty's stakes. Fix a row $i = 1, \dots, n$ and a column $j = 1, \dots, m$.

If $\eta_i(h_j) = 0$, i.e., if h_j does not occur in the possible world η_i , then the term $M_{i,j}$ in M coincides with $\beta(h_j)$. Recalling (2.2), Bookie's profit/loss is $\beta(h_j)s(h_j) \in \mathbb{R}$. On the other hand, if h_j occurs in the possible world η_i then $M_{i,j} = \beta(h_j) - 1$ and Bookie's profit/loss is $(\beta(h_j) - 1)s(h_j)$. In either case, Bookie's profit/loss for Betty's stake $s(h_j)$ on event h_j is $(\beta(h_j) - \eta_i(h_j))s(h_j)$ in the possible world η_i .

Let now $s \in \mathbb{R}^m$ be the (always column) vector whose coordinates are $s_1 = s(h_1), \dots, s_m = s(h_m)$. Then the vector $Ms \in \mathbb{R}^n$ gives the balance (2.3) in all possible worlds $\eta_1, \dots, \eta_n$. Bookie's β is inconsistent iff Betty can devise a vector $s \in \mathbb{R}^m$ (equivalently, $s \in \mathbb{Q}^m$) such that all coordinates of Ms are < 0 .

2.2 The Consistency (alias Dutch book) Theorem

Theorem 2.2. (De Finetti's consistency theorem, [1, pp. 309-313]) *For any boolean algebra A with atoms $a_1, \dots, a_n$ and corresponding homomorphisms*

² It is understood that $\eta_i(h_j)$ equals 1 or 0 according as h_j occurs or does not occur in η_i .

$\eta_1, \dots, \eta_n$ of A into $\{0, 1\}$, let $E = \{h_1, \dots, h_m\} \subseteq A$ and $\beta: E \rightarrow [0, 1]$. Then precisely one of the following conditions holds:

- (i) β is inconsistent in A in the sense of Definition 2.1. Thus there is a function $s: E \rightarrow \mathbb{R}$ such that, letting $s_1 = s(h_1), \dots, s_m = s(h_m)$, every coordinate of the vector $M(s_1, \dots, s_m) \in \mathbb{R}^n$ is < 0 . In symbols,

$$\sum_{j=1}^m s(h_j)(\beta(h_j) - \eta_i(h_j)) < 0 \text{ for every } i = 1, \dots, n.$$

- (ii) β is extendable to a state σ of A .

Stated otherwise, β is consistent in A iff it is extendable to a state of A .

Proof. In view of Theorem 1.8, by identifying A with the boolean algebra $\{0, 1\}^n$ of the vertices of the n -cube $[0, 1]^n$, each atom a_i is geometrically realized as the i th standard basis vector e_i in $\mathbb{R}^n$. Furthermore, each element h_j is realized as a vertex of the n -cube $[0, 1]^n$. Since the isomorphism ι of Theorem 1.8 now coincides with identity, by Corollary 1.9(ii) we can write

$$\eta_i = \theta_i = \theta_{e_i} \in \text{hom}(\{0, 1\}^n) \text{ for each } i = 1, \dots, n. \quad (2.5)$$

Proof that if condition (i) fails then condition (ii) holds. Suppose condition (i) fails. Gordan's theorem³ gives a nonzero column vector $(u_1, \dots, u_n) \in \mathbb{R}^n$ with each coordinate $u_i \geq 0$ and $u^T M = 0 =$ the origin in $\mathbb{R}^m$. Without loss of generality, $u_1 + \dots + u_n = 1$. Summing up,

$$u^T M = 0, \quad 0 \leq u_i \text{ } (i = 1, \dots, n), \quad u_1 + \dots + u_n = 1. \quad (2.6)$$

For each $j = 1, \dots, m$ the j th column C_j of the $n \times m$ matrix M in (2.4) has the form $C_j = C_{j,\text{left}} - C_{j,\text{right}}$, where

$$C_{j,\text{left}} = \underbrace{(\beta(h_j), \dots, \beta(h_j))}_{i \text{ times}}. \quad (2.7)$$

For each $i = 1, \dots, n$, the i th term $(C_{j,\text{right}})_i$ satisfies the identities

$$(C_{j,\text{right}})_i = \eta_i(h_j) = \theta_i(h_j) = e_i^T h_j = i\text{th coordinate of } h_j \in \{0, 1\}^n. \quad (2.8)$$

This follows from 1.8-1.9 and (2.5). By (2.6)-(2.8), the assumption $u^T M = 0$, (i.e., $u^T C_{j,\text{right}} = u^T C_{j,\text{left}}$ for each $j = 1, \dots, m$) amounts to writing

$$\beta(h_j) = u^T C_{j,\text{right}} = u^T h_j, \text{ for each } j = 1, \dots, m. \quad (2.9)$$

Let the state σ of the boolean algebra $\{0, 1\}^n$ be defined by

³ A self-contained proof of Gordan's theorem is given in 2.10.

$$\sigma(e_i) = u_i, \quad (i = 1, \dots, n).$$

By Corollary 1.11(iii) and (2.9), $\sigma(h_j) = u^T h_j = \beta(h_j)$ for each $j = 1, \dots, m$. This shows that σ extends β , whence condition (ii) holds.

Proof that conditions (i) and (ii) are incompatible. By way of contradiction, assume both conditions hold. As in Corollary 1.11, let the vector $v_\sigma \in \mathbf{S}$ be defined by $v_\sigma^T w = \sigma(w)$ for all $w \in \{0, 1\}^n$. In particular, $v_\sigma^T h_j = \sigma(h_j)$ for each $j = 1, \dots, m$. Since all coordinates of v_σ are ≥ 0 and their sum is equal to 1, then

$$v_\sigma^T C_{j,\text{left}} = \beta(h_j) \quad \text{for all } j = 1, \dots, m.$$

Further, by (2.8), $v_\sigma^T C_{j,\text{right}} = v_\sigma^T h_j = \sigma(h_j)$. Since, by condition (ii), σ extends β then

$$v_\sigma^T C_{j,\text{left}} = v_\sigma^T C_{j,\text{right}} \quad \text{for all } j = 1, \dots, m,$$

whence $v_\sigma^T M = 0$. As in Gordan's theorem 2.10, conditions (i) and (ii) imply the contradiction

$$0 \neq \underbrace{v_\sigma^T}_{v_\sigma \in \mathbf{S}} \underbrace{(M(s_1, \dots, s_m))}_{s_j < 0, \text{ for each } j=1, \dots, m} = (v_\sigma^T M)(s_1, \dots, s_m) = 0.$$

We have thus shown that conditions (i) and (ii) are incompatible.

Conclusion. If β is consistent then condition (i) fails, whence, by condition (ii), β is extendable to a state. Conversely, if condition (ii) holds then β cannot satisfy condition (i), i.e., β is consistent. $\square$

2.3 Interlude: Logical consistency and de Finetti's consistency

This section requires some acquaintance with the syntax and semantics of boolean propositional logic: formulas, truth-valuations, logical equivalence and consistency. The first pages of any book on mathematical logic will provide all necessary background. Readers not interested in the relationships between logical consistency and de Finetti's consistency can skip this section on a first reading, as no results proved here will be used in the rest.

Let $\mathbf{F}_n$ be the free boolean algebra over the free generating set $\{X_1, \dots, X_n\}$ introduced in Section 1.3. Elements of $\mathbf{F}_n$ are boolean formulas ϕ in the variables $X_1, \dots, X_n$ up to logical equivalence. The boolean operations naturally act on $\mathbf{F}_n$ as the boolean connectives act on formulas. Thus, e.g., let ϕ be a formula and $\tilde{\phi} \in \mathbf{F}_n$ the (infinite!) set of all formulas logically equivalent to ϕ . Then for any formula ψ logically equivalent to ϕ the equivalence class $\widetilde{\neg\psi}$ of $\neg\psi$ coincides with the complement $\neg\tilde{\phi} \in \mathbf{F}_n$ of the equivalence class $\tilde{\phi}$. Any formula ψ logically equivalent to ϕ is said to *code* the element $\tilde{\phi}$ of $\mathbf{F}_n$.

For each $i = 1, \dots, n$, X_i is a special kind of a formula, known as a *variable*. Traditionally, the same notation is used for X_i and for the equivalence class $\widehat{X_i} \in \mathbf{F}_n$. Likewise, $\neg, \wedge, \vee$ denote both the connectives acting on formulas, and the operations of the boolean algebra $\mathbf{F}_n$.

A *truth-valuation* is a $\{0, 1\}$ -valued function v defined on all formulas $\psi = \psi(X_1, \dots, X_n)$, having the following properties:

$$v(\neg\psi) = 1 - v(\psi), \quad v(\psi \wedge \phi) = \min(v(\psi), v(\phi)), \quad v(\psi \vee \phi) = \max(v(\psi), v(\phi)).$$

The non-ambiguity of the syntax ensures that v is uniquely determined by the values it assigns to the variables $X_1, \dots, X_n$.

Let the formulas $\phi_1, \dots, \phi_m$ respectively code elements $h_1, \dots, h_m$ of $\mathbf{F}_n$. Then the set $\{\phi_1, \dots, \phi_m\}$ is said to be *logically consistent* if some truth-valuation assigns value 1 to each formula ϕ_i . In equivalent algebraic terms, $\eta(h_1) = \dots = \eta(h_m) = 1$ for some $\eta \in \text{hom}(\mathbf{F}_n)$.

Corollary 2.3. *Let $E = \{h_1, \dots, h_m\}$ be a finite subset of the boolean algebra $\mathbf{F}_n$. Let the function $\beta: E \rightarrow \{0, 1\}$ assign the constant value 1 to each element of E . Then the following conditions are equivalent:*

- (i) E can be coded by a logically consistent set E' of boolean formulas.
- (ii) β is (de Finetti) consistent in the sense of Definition 2.1.
- (iii) β can be extended to a homomorphism of $\mathbf{F}_n$ into $\{0, 1\}$.
- (iv) β can be extended to a state of $\mathbf{F}_n$.

Proof. (ii) $\Leftrightarrow$ (iv) By Theorem 2.2.

(i) $\Rightarrow$ (iii) By hypothesis, some truth-valuation v assigns 1 to all formulas in E' . Equivalently, $\eta(h_1) = \dots = \eta(h_m) = 1$ for some $\eta \in \text{hom}(\mathbf{F}_n)$. Thus, η extends β .

(iii) $\Rightarrow$ (ii) Every homomorphism of $\mathbf{F}_n$ into $\{0, 1\}$ is a state. Now apply Theorem 2.2.

(ii) $\Rightarrow$ (i) Assume (i) does not hold, with the intent of proving that β is inconsistent in the sense of Definition 2.1. To this purpose, for each $j = 1, \dots, m$, Betty places the stake $s(h_j) = -1$. As a result, she now receives m from Bookie. In any possible world $\eta \in \text{hom}(\mathbf{F}_n)$ she will have to return 1 to Bookie for each event to which η assigns the value 1. However, at least one event $h = h_\eta \in E$ will be evaluated 0 by η , because the assumed logical inconsistency of E' entails that no truth-valuation v assigns value 1 to all formulas of E' . Thus Betty has a net profit ≥ 1 in any possible world η . By Definition 2.1, β is inconsistent. $\square$

Using the fact that every boolean algebra A is the homomorphic image of some free boolean algebra, with some extra work Corollary 2.3 can be shown to hold for A .

2.4 De Finetti's Consistency Theorem for all boolean algebras

Generalizing Theorem 2.2, in Theorem 2.7 we will show that our restriction to finite boolean algebras is inessential. So let us relax this restriction:

A henceforth stands for an arbitrary (finite or infinite) boolean algebra.

The set $\text{hom}(A)$ of homomorphisms of A into the two-element boolean algebra $\{0, 1\}$ is no longer indexed by the atoms of A . Suffice to say that an infinite boolean algebra A need not have atoms. Thus the definition of consistency gets the following form, which is equivalent to Definition 2.1 for finite boolean algebras, and encompasses the general case when the boolean algebra A is infinite:

Definition 2.4. Let $E = \{h_1, \dots, h_m\}$ be a subset of a boolean algebra A and $\beta: E \rightarrow [0, 1]$ a function. Then β is said to be *inconsistent in A* if there is a function $s: E \rightarrow \mathbb{R}$ such that

$$\sum_{j=1}^m s(h_j)(\beta(h_j) - \eta(h_j)) < 0 \quad \text{for every } \eta \in \text{hom}(A). \quad (2.10)$$

If β is not inconsistent in A we say it is *consistent in A* .

The definition of a *state* of A is verbatim the same as Definition 1.10.

Lemma 2.5. Let B be a finite boolean algebra and B' a subalgebra of B .

- (i) The homomorphisms of B' into $\{0, 1\}$ are precisely the restrictions to B' of the homomorphisms of B into $\{0, 1\}$.
- (ii) The states of B' are precisely the restrictions to B' of the states of B .

Proof. (i) Let $\epsilon \in \text{hom}(B)$. Trivially, the restriction $\epsilon|_{B'}$ of ϵ to B' is a homomorphism of B' into $\{0, 1\}$.

Conversely, let $\eta \in \text{hom}(B')$, with the intent of extending η to a some $\eta^* \in \text{hom}(B)$. By Corollary 1.9(i) we may write $\eta = \eta_e$ for a unique atom e of B' . By Proposition 1.5, each atom a of B' is a nonzero element of B , and dominates a nonempty set $[a]$ of atoms of B . If $a \neq b \in \text{at}(B')$ then $[a] \cap [b] = \emptyset$. (For otherwise, some atom c of B is dominated by both a and b , whence c is dominated by $a \wedge b = 0$, which is impossible.) As a consequence, upon writing

$$c \approx d \quad \text{iff } c \text{ and } d \text{ are dominated by the same atom } a \text{ of } B', (c, d \in \text{at}(B)),$$

we obtain an equivalence relation $\approx$ over $\text{at}(B)$. The function $a \mapsto [a]$, $a \in \text{at}(B')$ maps $\text{at}(B')$ onto the set of $\approx$ -equivalence classes. Furthermore, for

each $x \in B'$ letting $a_1, \dots, a_m$ be the atoms of B' dominated by x , we have the identity

$$x = \bigvee \{b \in \mathbf{at}(B) \mid b \in ([a_1] \vee \dots \vee [a_m])\}. \quad (2.11)$$

Turning to our homomorphism $\eta_e \in \mathbf{hom}(B')$, let us arbitrarily pick an atom $e^* \in [e]$, and let $\eta_{e^*} \in \mathbf{hom}(B)$ be the corresponding homomorphism. By a final application of Corollary 1.9, for every $y \in B$, $\eta_{e^*}(y) = 1$ iff y dominates e^* . An easy verification using (2.11) shows that the homomorphism $\eta^* = \eta_{e^*}$ is an extension of η_e .

(ii) The restriction to B' of any state of B is a state of B' . Conversely, let τ be a state of B' . With the notation of (i), for each $a \in \mathbf{at}(B')$ arbitrarily pick an atom $a^* \in [a] \subseteq \mathbf{at}(B)$. Let the function $\tau^\circ: \mathbf{at}(B) \rightarrow \{0, 1\}$ be defined by

$$\tau^\circ(a^*) = \tau(a) \text{ for each } a \in \mathbf{at}(B'), \text{ and } \tau^\circ(b) = 0 \text{ for all other atoms of } B.$$

By Corollary 1.11(iii), any state of the finite boolean algebra B is uniquely determined by the values it gives to the atoms of B . It is now easy to verify that τ° uniquely determines a state τ^* of B which extends τ . $\square$

Lemma 2.6. *Let B' be a finite subalgebra of an infinite boolean algebra A .*

(i) *The states of B' are precisely the restrictions to B' of the states of A .*

(ii) *The homomorphisms of B' into $\{0, 1\}$ are precisely the restrictions to B' of the homomorphisms of A into $\{0, 1\}$.*

Proof. (i) Let σ be a state of B' . By Lemma 2.5(ii), for every finite subalgebra B of A containing B' , σ is extendable to a state of B . Let the set $\tilde{S}_{\sigma, B} \subseteq [0, 1]^A$ be defined by

$$\tilde{S}_{\sigma, B} = \{f: A \rightarrow [0, 1] \mid f \upharpoonright B \text{ is a state of } B \text{ extending } \sigma\}.$$

By definition of the product topology of $[0, 1]^A$, $\tilde{S}_{\sigma, B}$ is a nonempty *closed* subset of the compact Hausdorff space $[0, 1]^A$.

For any finite family of finite subalgebras $B_1, \dots, B_u$ of A containing B' the intersection I of the closed sets $\tilde{S}_{\sigma, B_1}, \dots, \tilde{S}_{\sigma, B_u}$ is nonempty. As a matter of fact, letting B_{u+1} be the subalgebra of A generated by the finite set $B_1 \cup \dots \cup B_u$, it follows that B_{u+1} is finite (see Section 1.3), and the set I contains the nonempty set $\tilde{S}_{\sigma, B_{u+1}}$. Next let

$$I^* = \bigcap \{\tilde{S}_{\sigma, B} \mid B \text{ a finite subalgebra of } A \text{ containing } B'\}.$$

From the compactness of the Tychonoff cube $[0, 1]^A$ it follows that I^* is nonempty. Any element of I^* is a state of A extending σ .

(ii) A routine variant of the proof of (i) using Lemma 2.5(i). $\square$

Theorem 2.2 has the following generalization to every (finite or infinite) boolean algebra A :

Theorem 2.7. (De Finetti consistency theorem, general case) *Let E be a finite subset of a boolean algebra A and $\beta: E \rightarrow [0, 1]$ a function. Then β is consistent in A iff β is extendable to a state of A .*

Proof. Say $E = \{h_1, \dots, h_m\}$. The subalgebra B_E of A generated by E is finite. (As noted in Section 1.3, the number of elements of B_E is $\leq 2^m$.)

($\Rightarrow$) Let $s: E \rightarrow \mathbb{R}$ be Betty's bet on the events in E . Let us agree to say that any $\eta = \eta_s \in \text{hom}(B_E)$ such that

$$\sum_{l=1}^m s(h_l)(\beta(h_l) - \eta_s(h_l)) \geq 0 \quad (2.12)$$

is a *witness of the s -consistency of β in B_E* . Since, by assumption, β is consistent in A , for every bet $t: E \rightarrow \mathbb{R}$ there is a witness $\theta = \theta_t \in \text{hom}(A)$ of the t -consistency of β in A .⁴ A fortiori, the restriction of θ_t to B_E witnesses the t -consistency of β in B_E . Since t is any arbitrary bet on E , β is consistent in B_E . By Theorem 2.2, β is extendable to a state τ of B_E . By Lemma 2.6(i), τ is extendable to a state ρ of A . A fortiori, $\beta \subseteq \tau$ is extendable to ρ .

($\Leftarrow$) Let σ be a state of A extending β . The restriction $\sigma \upharpoonright B_E$ is a state of B_E extending β . By Theorem 2.2, β is consistent in B_E . We have to prove that β is consistent in A . Arguing by way of contradiction, let us assume Betty can devise a bet $s: E \rightarrow [0, 1]$ such that

$$\sum_{j=1}^m s(h_j)(\beta(h_j) - \eta(h_j)) < 0 \quad \text{for every } \eta \in \text{hom}(A).$$

By Lemma 2.6(ii), $\sum_{j=1}^m s(h_j)(\beta(h_j) - \epsilon(h_j)) < 0$ for every $\epsilon \in \text{hom}(B_E)$. By Definition 2.4, β is inconsistent in B_E , a contradiction. $\square$

Corollary 2.8. *Let A be a boolean algebra. Then A has a state. A function $f: A \rightarrow [0, 1]$ is a state of A iff f is finitely consistent, in the sense that every finite restriction of f is consistent in A .*

Remarks 2.9 (i) The proof of Lemma 2.6 rests on the compactness of the Tychonoff cube $[0, 1]^A$. In ZF (Zermelo-Fraenkel) set-theory this is an equivalent reformulation of the Axiom of Choice. The dependence of de Finetti's consistency theorem 2.7 on this axiom is rarely made explicit in the vast literature on this theorem.

(ii) From Lemmas 2.5-2.6 and Theorem 2.7 it follows that a function $\beta: E \rightarrow [0, 1]$ is consistent in a boolean algebra $A \supseteq E$ iff it is consistent

⁴ The definition of $\theta_t \in \text{hom}(A)$ witnessing the t -consistency of β in A is precisely the same as (2.12), with t in place of s .

in the algebra B_E generated by E in A , iff it is consistent in any algebra A^* containing A as a subalgebra. Hence the consistency of β is largely indifferent to the chosen boolean algebra $A \supseteq E$. However, the specification of *some* ambient boolean algebra A for the set E of “events” and their operations considered in de Finetti's consistency theorem is necessary to give a meaning to expressions such as “in any possible case”, or “in any possible world”, which occur in the definition of consistency.

(iii) De Finetti's consistency theorem has two directions. He proved the $(\Rightarrow)$ -direction in [1, pp.309-312], and the $(\Leftarrow)$ -direction in [1, p.313]. Nevertheless, purported “converses” of de Finetti consistency ($\sim$ Dutch book) theorem exist in the literature.

2.5 A self-contained proof of Gordan's Theorem

In this section we give a self-contained proof of Gordan's theorem, a basic ingredient of the proof of de Finetti's consistency theorem.

Theorem 2.10. (Gordan's theorem) *Let M be a real matrix with n rows and m columns. Then precisely one of the following conditions holds:*

- (i) *There is a (column) vector $s \in \mathbb{R}^m$ such that every coordinate of the vector $Ms \in \mathbb{R}^n$ is < 0 .*
- (ii) *There is a nonzero vector $u \in \mathbb{R}^n$ such that every coordinate of u is ≥ 0 and $u^T M = 0 =$ the origin in $\mathbb{R}^m$. (As above, u^T is the transpose of u .)*

Proof. Conditions (i) and (ii) in the statement of Gordan's theorem 2.10 are promptly seen to be incompatible. For, if both are assumed to hold we have the contradiction

$$0 \neq \underbrace{u^T}_{\geq 0 \dots \geq 0, u \neq 0} \underbrace{(Ms)}_{< 0 \dots < 0} = (u^T M) s = 0. \quad (2.13)$$

There remains to be proved that if (i) fails then (ii) holds. Failure of (i) means that the range R of the linear operator $M: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is disjoint from the south-west open octant O given by

$$O = \{v \in \mathbb{R}^n \mid \text{each coordinate of } v \text{ is } < 0\}.$$

R is a linear subspace of $\mathbb{R}^n$ containing the origin. The hyperplane separation lemma 2.14 ⁵ then yields a nonzero column vector $u \in \mathbb{R}^n$, along with a hyperplane H such that

$$R \subseteq H = \{x \in \mathbb{R}^n \mid u^T x = 0\} \quad \text{and} \quad O \subseteq \{x \in \mathbb{R}^n \mid u^T x < 0\}.$$

⁵ A self-contained proof is given below.

The latter inclusion implies that all coordinates of u are ≥ 0 . It is now easy to see that $u^T M = 0$. For otherwise, (absurdum hypothesis), there is a vector $q \in \mathbb{R}^m$ with $(u^T M)q \neq 0$. As a member of R , the vector Mq is orthogonal to u^T , whence $u^T(Mq) = 0$, a contradiction. We have thus proved that if (i) fails then (ii) holds. $\square$

A proof of the hyperplane separation lemma 2.14

To complete the proof of Gordan's theorem, as well as of de Finetti's theorems 2.2 and 2.7, in this section a routine proof is given of the hyperplane separation lemma 2.14.⁶

Fact 2.11 *The closure $\text{cl}X$ of a convex set $X \subseteq \mathbb{R}^n$ is convex.*

Proof. If $X = \emptyset$ we are done. If $X \neq \emptyset$, for any $x, y \in \text{cl}X$ we must prove that the segment $[x, y] = \{(1 - \lambda)x + \lambda y \mid \lambda \in [0, 1]\}$ lies in $\text{cl}X$. By definition of closure, X contains converging sequences $x_n \rightarrow x$ and $y_n \rightarrow y$. Since X is convex, the interval $[x_n, y_n]$ lies in X . The sequence of midpoints m_n of $[x_n, y_n]$ converges to the midpoint m of $[x, y]$. As a limit of a convergent sequence of points of X , m lies in $\text{cl}X$. More generally, for each $\lambda \in [0, 1]$ the sequence $(1 - \lambda)x_n + \lambda y_n$ converges to $(1 - \lambda)x + \lambda y$. So $(1 - \lambda)x + \lambda y$ lies in $\text{cl}X$. As λ ranges over $[0, 1]$, all points of $[x, y]$ are obtained. They all lie in $\text{cl}X$. $\square$

Let us recall that the scalar product of two (always column) vectors $u, v \in \mathbb{R}^n$ is given by matrix multiplication $u^T v$, with u^T the transpose of u .

For $w \in \mathbb{R}^n$ we let $|w|$ denote its euclidean norm, (or "length") $(w^T w)^{1/2}$.

Fact 2.12 *Let $X \subseteq \mathbb{R}^n$ be a nonempty closed convex set.*

(i) *X has a unique shortest vector, i.e., a vector x_* with $|x_*| = \inf\{|x| \mid x \in X\}$.*

(ii) *Let $t \in \mathbb{R}^n$. Then there is a unique point u in X such that $|u - t| = \inf\{|x - t| \mid x \in X\}$.*

(iii) *Assume $0 \notin X$. Let x_* be the shortest vector in X given by (i). Then for all $y \in X$, $x_*^T y > |x_*|^2/2$.*

(iv) *Let x_* be the shortest vector in X . Assume $0 \notin X$. Then the hyperplane*

$$H = \{z \in \mathbb{R}^n \mid x_*^T z = |x_*|^2/2\}$$

⁶ Some proofs of de Finetti's consistency theorem in the literature cite a "hyperplane separation theorem" without indicating the specific result needed for the proof.

strongly separates the origin from X .⁷ More generally, for any nonempty closed convex set $Y \subseteq \mathbb{R}^n$, point $e \in \mathbb{R}^n \setminus Y$, letting x_* be the unique point of Y closest to e , we have $(x_* - e)^T(x - e) > |x_* - e|^2/2$ for each $x \in Y$. In other words, the hyperplane

$$K = \{z \in \mathbb{R}^n \mid (x_* - e)^T(z - e) = |x_* - e|^2/2\}$$

strongly separates e from K .

Proof. (i) If the origin 0 belongs to X we have nothing to prove. Otherwise, let ξ be shorthand for $\inf\{|x| \mid x \in X\}$. Since X is nonempty, it is no loss of generality to assume that X is bounded. By definition of infimum there is a sequence $x_n \in X$ with $|x_n| - \xi < 1/n$. There is a convergent subsequence x_{n_i} . The point $x_* = \lim x_{n_i}$ belongs to X , because X is closed. Since $x \mapsto |x|$ is a continuous real-valued function, then $\xi = \lim |x_{n_i}| = |\lim x_{n_i}| = |x_*|$. To prove the uniqueness of x_* suppose (absurdum hypothesis) $y \in X$ is different from x_* and has the same length as x_* . The triangle with vertices $0, x_*, y$ is isosceles. The midpoint of the interval $[x_*, y]$ has a distance $< |x_*|$ from 0, which is impossible.

(ii) The translated set $X - t = \{x - t \mid x \in X\}$ is also nonempty closed and convex. Distances $|a - b|$ are preserved under translation. Now apply (i).

(iii) Arguing by way of contradiction, let us assume that there exists $y \in X$ such that

$$x_*^T y \leq |x_*|^2/2. \quad (2.14)$$

The closed interval $[x_*, y] = \{(1 - \lambda)x_* + \lambda y \mid \lambda \in [0, 1]\}$ is contained in X . For all $0 \leq \lambda \leq 1$ we then have:

$$\begin{aligned} |x_*|^2 &\leq |(1 - \lambda)x_* + \lambda y|^2, \text{ since } x_* \text{ is the shortest vector in } X \\ &= (1 - \lambda)^2|x_*|^2 + \lambda^2|y|^2 + 2(1 - \lambda) \cdot \lambda x_*^T y \\ &= |x_*|^2 + \lambda^2|x_*|^2 - 2\lambda|x_*|^2 + \lambda^2|y|^2 + (1 - \lambda)\lambda|x_*|^2, \text{ by (2.14).} \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\leq \lambda^2|x_*|^2 - 2\lambda|x_*|^2 + \lambda^2|y|^2 + \lambda|x_*|^2 - \lambda^2|x_*|^2 \\ &= -\lambda|x_*|^2 + \lambda^2|y|^2. \end{aligned}$$

We then have $|x_*|^2 \leq \lambda|y|^2$ for each $\lambda > 0$. The only possibility is $|x_*|^2 = 0$ whence x_* coincides with the origin of $\mathbb{R}^n$, a contradiction.

(iv) Immediate from (iii). □

⁷ i.e., the distance of the origin from H is > 0 , the distance from H of every $y \in X$ is > 0 , and X and the origin are contained in opposite closed half-spaces with boundary H .

Fact 2.13 (Supporting hyperplane theorem) *Let X be a closed convex subset of $\mathbb{R}^n$ having a boundary point b . Then there is a hyperplane H containing b , such that X is contained in one of the two closed half-spaces bounded by H . H is known as a supporting hyperplane for X at b .*

Proof. By definition of boundary, b lies in the closed set X , and there is a sequence of points $b_n \notin X$ with $b_n \rightarrow b$. For each b_n , let $x_n \in X$ be the closest point to b_n as given by Fact 2.12(ii). Let u_n be the unit vector $(x_n - b_n)/|x_n - b_n|$. By Fact 2.12(iv),

$$\frac{|x_n - b_n|}{2} < u_n^T(x - b_n) \quad \text{for each } x \in X, \text{ in particular for } x = b. \quad (2.15)$$

The bounded sequence u_n has a convergent subsequence, say converging to the unit vector u . Then by (2.15),

$$\lim \frac{|x_n - b_n|}{2} \leq \lim u_n^T(b - b_n) = u^T \lim(b - b_n) = 0,$$

whence

$$\lim \frac{|x_n - b_n|}{2} = 0. \quad (2.16)$$

From (2.15)-(2.16), for each $x \in X$ we have

$$0 \leq \lim u_n^T(x - b_n) = u^T(x - b) \quad \text{whence } u^T b \leq u^T x.$$

Thus the hyperplane $H = \{x \in \mathbb{R}^n \mid u^T x = u^T b\}$ supports X at b . $\square$

We are now ready to prove the specific hyperplane separation lemma needed for the proof of Gordan's theorem and, ultimately, for the proof of de Finetti's consistency theorem.

Lemma 2.14. (Hyperplane Separation Theorem) *Let $O \subseteq \mathbb{R}^n$ be the set of all vectors whose coordinates are < 0 . We say that O is the south-west open octant. Let R be a linear subspace of $\mathbb{R}^n$ disjoint from O . Then for some vector $u \in \mathbb{R}^n$ the hyperplane $H = \{x \in \mathbb{R}^n \mid u^T x = 0\}$ has the following (separation) property for O and R :*

$$R \subseteq H, \quad (\text{i.e., } u^T r = 0 \text{ for all } r \in R).$$

and

$$u^T y < 0 \text{ for all } y \in O, \text{ whence } H \cap O \text{ is empty.}$$

Proof. The set $R + O = \{r + o \in \mathbb{R}^n \mid r \in R, o \in O\}$ is convex and does not contain the origin. Its closure $\text{cl}(R + O)$ is convex, by Fact 2.11. Since R and O are disjoint, the origin is a boundary point of $\text{cl}(R + O)$.

Fact 2.13 provides a supporting hyperplane H for $\text{cl}(R + O)$ at 0. Since R is linear and $0 \in R \cap H$ then R is contained in H ,

$$R \subseteq H. \quad (2.17)$$

By construction, O is contained in one of the two *open* half-spaces with boundary H . The other open half-space contains a vector u orthogonal to H such that $u^T y < 0$ for all $y \in O$. For any such u , by (2.17), we automatically have $u^T r = 0$ for all $r \in R$. $\square$

The proof of Gordan's theorem is now complete, and so is the proof of de Finetti's consistency theorem 2.2, as well as of its generalization 2.7 to all boolean algebras.

Remarks 2.15 The main effect of de Finetti's consistency theorem is summarized by de Finetti himself in his quote at the beginning of these handouts: all the results of probability theory are nothing more than consequences of his definition of consistency. In particular, the traditional "*axiom* of additivity for the probability of incompatible events" is shown by Theorem 2.2 to be a *consequence of de Finetti's definition* 2.1 of consistency. In a nutshell:

$\text{consistency} + \text{incompatibility} \Rightarrow \text{additivity axiom}$	(2.18)
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Later on, we will see that Theorem 2.2 has a key role in deriving the "product law" for logically-algebraic independent events. Furthermore, the finite additivity of states will be reconciled with the countable additivity of the probability measures in Kolmogorov's axiomatic theory. See, in particular, (4.1).

References

1. B. de Finetti, Sul significato soggettivo della probabilità, *Fundamenta Mathematicae*, 17 (1931) 298–329. Translated into English as: On the Subjective Meaning of Probability. In: P. Monari, D. Cocchi (Eds.), *Probabilità e Induzione*, Clueb, Bologna, pp. 291–321, 1993. Reprinted in [2, pp. 191–222].
2. B. de Finetti, *Opere Scelte*. (Selected Works), Vol. 1. Unione Matematica Italiana, Edizioni Cremonese, Firenze, 2006.

Independence: Deriving the Product Law

There is, thus, independence in a vague and intuitive sense, and there is “independence” in the narrow but well-defined sense that the rule of multiplication of probabilities is applicable. It was the vague and intuitive notions that provided for a long time the main motivation and driving force behind probability theory. And while an impressive formalism was being created, mathematicians (with very few exceptions) remained aloof because it was not clear to them what the objects were to which the formalism was applicable.

Page 10 in: M. Kac, “Statistical independence in probability, analysis and number theory.” The Carus Mathematical Monographs, Vol. 12, The Mathematical Association of America. J.Wiley and Sons, Inc., New York (1959).

A definition of the vague and intuitive notion of independence exists in logic and algebra. Combining this rigorous notion with de Finetti’s consistency theorem, in this chapter we will *derive* the rule of multiplication of probabilities for independent events.

3.1 Algebraic Independence

In probability textbooks, two sets A and B of events are said to be (stochastically) independent iff they obey the “product law”, or “rule of multiplication”,

$$P(a \wedge b) = P(a) \cdot P(b) \text{ for all } a \in A \text{ and } b \in B.$$

First of all, one may wonder why the independence of A and B , an essentially logic-algebraic notion, should assume that A and B are assigned values by some function P . In fact, the outcome of any football match is independent of the outcome of any horse race. No quantitative estimation is involved in this intuition of independence.

Second, of all the possible operations that calculate the probability of a conjunction of “independent” events, what is so special about multiplication?

We will answer both questions in two steps. First, Theorem 3.3 provides an elementary proof that *if there exists* a consistency preserving continuous operation $\odot$ for the probability $P(a) \odot P(b)$ of the conjunction $P(a \wedge b)$ of events belonging to two books in algebraically independent sets, *then multiplication is the only possible candidate* for such an operation.

Next, the less elementary Theorem 3.8 shows that multiplication does satisfy (and therefore is characterized by) this condition. In more detail, multiplication is the only continuous $[0, 1]$ -valued operation on $[0, 1]^2$ that preserves the consistency of *any* book of conjunctions of algebraically independent events—e.g., a book of conjunctions of events taken from two books: one on horse racing and the other on football match scores.

So bookmakers who do not comply with the product law for independent events are warned that they may run into inconsistency.¹

Independent boolean algebras

Generalizing (1.4), for any (finite or infinite) set $X \neq \emptyset$ we let $\mathbf{pow}(X)$ be the powerset of X , equipped with union, intersection and complement. This is a boolean algebra with $2^{|X|}$ elements, $|X|$ denoting the cardinality of X .

If X is finite, the set $\mathbf{at}(\mathbf{pow}(X))$ of atoms of the boolean algebra $\mathbf{pow}(X)$ is in one-one correspondence with X , and also with the set $\mathbf{hom}(\mathbf{pow}(X))$ of homomorphism of the boolean algebra $\mathbf{pow}(X)$ into the two-element boolean algebra $\{0, 1\}$. Furthermore, Proposition 1.5 shows that any finite boolean algebra is isomorphic to $\mathbf{pow}(Y)$ for some finite set Y .

More generally, for any boolean algebra A , the set $\mathbf{hom}(A)$ conveniently replaces the set of atoms of A .

In combination with de Finetti’s consistency, the following algebraic notion of independence is fundamental for our analysis of stochastic independence:

Definition 3.1. (See, e.g., [2, Chapter 4, 11.3]) Two subalgebras A, B of a boolean algebra C are said to be *independent* (in C) if every pair $(a, b) \in A \times B$ with $a \neq 0$ and $b \neq 0$ satisfies $a \wedge b \neq 0$.

Intuitively, the occurrence or non-occurrence of any event in A and B has no effect on the occurrence or non-occurrence of any event in C .

Example 3.2. By Proposition 1.5, for A and B finite boolean algebras, one may write

$$A = \mathbf{pow}(X) \quad \text{and} \quad B = \mathbf{pow}(Y) \quad \text{for finite sets } X, Y. \quad (3.1)$$

¹ This is not a problem for real-life bookmakers, who traditionally have little inclination to trade their roles with bettors.

It is no loss of generality to assume that X and Y are nonempty finite sets of integers > 0 . Thus one may list X along the x -axis of the cartesian plane, and Y along the y -axis. Cylindrification along the y -axis transforms every subset S of X into the subset $S^\uparrow$ of $\mathbf{pow}(X \times Y)$ given by

$$S^\uparrow = \{(x, y) \in \mathbf{pow}(X \times Y) \mid x \in S, y \in Y\}.$$

Symmetrically, cylindrification along the x -axis transforms every $T \subseteq Y$ into

$$T^\rightarrow = \{(x, y) \in \mathbf{pow}(X \times Y) \mid x \in X, y \in T\}.$$

Letting S and T range over arbitrary subsets of X and of Y , the cylindrification maps $S \mapsto S^\uparrow$ and $T \mapsto T^\rightarrow$ yield one-one homomorphisms (embeddings), respectively of the boolean algebras $\mathbf{pow}(X)$ and $\mathbf{pow}(Y)$ into $\mathbf{pow}(X \times Y)$.

Recalling (3.1), let $A^\uparrow$ and $B^\rightarrow$ denote the images of A and B under these embeddings:

$$A^\uparrow = \{S^\uparrow \mid S \subseteq X\}, \quad B^\rightarrow = \{T^\rightarrow \mid T \subseteq Y\}.$$

We claim that $A^\uparrow$ and $B^\rightarrow$ are independent subalgebras of $\mathbf{pow}(X \times Y)$, and the finite set $A^\uparrow \cup B^\rightarrow$ generates $\mathbf{pow}(X \times Y)$.

As a matter of fact, let us assume $0 \neq S^\uparrow \in A^\uparrow$ and $0 \neq T^\rightarrow \in B^\rightarrow$. This means that for some $x^* \in S$, $S^\uparrow \supseteq \{(x^*, y) \mid y \in Y\} \neq \emptyset$. Likewise, for some $y^* \in T$, $T^\rightarrow \supseteq \{(x, y^*) \mid x \in X\} \neq \emptyset$. Then the pair (x^*, y^*) belongs to $S^\uparrow \cap T^\rightarrow$. This shows that $A^\uparrow$ and $B^\rightarrow$ are independent subalgebras of $\mathbf{pow}(X \times Y)$. The set $A^\uparrow \cup B^\rightarrow$ generates $\mathbf{pow}(X \times Y)$ because already the cylindrifications $x^\uparrow$ of points of X and $y^\rightarrow$ of points of Y generate the finite set $\mathbf{pow}(X \times Y)$. Our claim is settled.

In conclusion, for all nonempty finite sets X and Y , cylindrification naturally identifies $X \times Y$ with the set of atoms of $\mathbf{pow}(X \times Y)$.

Corollary 3.5 will provide a generalization of this example to all boolean algebras A, B .

3.2 Consistency preserving books on independent sets of events

As usual, for any function f , $\mathbf{dom}(f)$ denotes the domain of definition of f .

Let $\odot$ be a continuous $[0, 1]$ -valued function defined over $[0, 1]^2$. Let A and B be independent subalgebras of a boolean algebra C . Suppose α and β are consistent books, respectively defined on the finite sets $\mathbf{dom}(\alpha) \subseteq A$ and $\mathbf{dom}(\beta) \subseteq B$. Let the function $\gamma_\times : E \times F \rightarrow [0, 1]$ be defined by

$$\gamma_\times(a, b) = \alpha(a) \odot \beta(b), \text{ for all } a \in E \text{ and } b \in F.$$

Let the finite set $C_\wedge$ be defined by

$$C_\wedge = \{c \in C \mid c = a \wedge b, \text{ for some } a \in E \text{ and } b \in F\}.$$

Next let us suppose for all $a, a' \in E$ and $b, b' \in F$

$$\text{if } a \wedge b = a' \wedge b' \text{ then } \alpha(a) \odot \beta(b) = \alpha(a') \odot \beta(b'). \quad (3.2)$$

Then upon writing $\gamma(a \wedge b) = \alpha(a) \odot \beta(b)$, ($a \in E$ and $b \in F$), we obtain from $\gamma_\times$ a (well defined) function $\gamma : C_\wedge \rightarrow [0, 1]$. We say that γ is the $\odot$ -book of α and β , and we denote it by $\alpha \odot \beta$.

Theorem 3.3. *Let $\odot : [0, 1]^2 \rightarrow [0, 1]$ be a continuous operation on $[0, 1]^2$ having the following properties:*

(i) *For all (finite or infinite) boolean algebras A, B, C with A and B independent subalgebras of C , and consistent books α, β respectively defined on finite subsets $\text{dom}(\alpha) \subseteq A$ and $\text{dom}(\beta) \subseteq B$, condition (3.2) holds, whence the $\odot$ -book $\alpha \odot \beta$ exists.*

(ii) *$\alpha \odot \beta$ is consistent.*

Then $\odot$ coincides with product.

Proof. For $M, N \in \{1, 2, \dots\}$ fixed but otherwise arbitrary integers, let the boolean algebras A, B, C and the consistent books α, β satisfy the following special conditions:

$$\text{dom}(\alpha) = A = \{a \times \{1, \dots, N\} \mid a \in \text{pow}(\{1, \dots, M\})\}, \quad (3.3)$$

$$\text{dom}(\beta) = B = \{\{1, \dots, M\} \times b \mid b \in \text{pow}(\{1, \dots, N\})\}, \quad (3.4)$$

$$C = \text{pow}(\{1, \dots, M\} \times \{1, \dots, N\}). \quad (3.5)$$

As in Example 3.2, A and B are independent subalgebras of C . Further, the set $A \cup B$ generates C . Let

$$\alpha \text{ be the book on } A \text{ assigning the value } 1/M \text{ to each atom of } A \quad (3.6)$$

and

$$\beta \text{ be the book on } B \text{ assigning the value } 1/N \text{ to each atom of } B. \quad (3.7)$$

It is sufficient to show that if the operation $\odot$ satisfies conditions (i)-(ii) for the class of finite boolean algebras A, B, C and consistent books α, β satisfying (3.3)-(3.7) then $\odot$ coincides with product.

To this purpose, let us preliminarily note that α and β are consistent books on $\text{dom}(\alpha)$ and $\text{dom}(\beta)$ respectively. This follows from de Finetti's consistency theorem 2.2 and (3.6)-(3.7), because there are (uniquely defined) states $\bar{\alpha}$ of A and $\bar{\beta}$ of B extending α and β .

Since $\odot$ satisfies condition (3.2) in (i), upon setting

$$\gamma(a \wedge b) = \alpha(a) \odot \beta(b) \text{ for all } a \in \text{dom}(\alpha) \text{ and } b \in \text{dom}(\beta),$$

we have a book $\gamma = \alpha \odot \beta$ defined on the finite set

$$C_\wedge = \{c \in C \mid c = a \wedge b, \text{ for some } a \in \text{dom}(\alpha) \text{ and } b \in \text{dom}(\beta)\}.$$

Since $\text{dom}(\alpha) = A$ and $\text{dom}(\beta) = B$, the set $\text{dom}(\alpha) \cup \text{dom}(\beta) = A \cup B$ is contained in $C_\wedge$, whence γ jointly extends α and β .

Since $\odot$ satisfies condition (ii), γ is a consistent book with $\text{dom}(\gamma) = C_\wedge \subseteq C$. De Finetti's consistency theorem 2.2 yields a state $\bar{\gamma}$ of C extending γ .

Next let us fix an atom $q \in \text{at}(B) \subseteq B$. For every element $a \in A$ we have:

$$\begin{aligned} \alpha(a) &= \bar{\gamma}(a), \text{ because } \bar{\gamma} \text{ agrees with } \alpha \text{ over } A \subseteq \text{dom}(\gamma) \\ &= \bar{\gamma}((a \wedge q) \vee (a \wedge \neg q)), \text{ distributivity of } A \\ &= \bar{\gamma}(a \wedge q) + \bar{\gamma}(a \wedge \neg q), \bar{\gamma} \text{ is additive on incompatible elements of } C_\wedge \\ &= \alpha(a) \odot \beta(q) + \bar{\gamma}(a \wedge \neg q), \text{ as } \bar{\gamma} \supseteq \gamma = \alpha \odot \beta \\ &= \alpha(a) \odot \frac{1}{N} + \bar{\gamma}(a \wedge \neg q), \text{ definition (3.7) of } \beta \\ &= \alpha(a) \odot \frac{1}{N} + \bar{\gamma} \left(a \wedge \bigvee_{z \in \text{at}(B), z \neq q} z \right), \text{ Proposition 1.5} \\ &= \alpha(a) \odot \frac{1}{N} + \bar{\gamma} \left(\bigvee_{z \in \text{at}(B), z \neq q} a \wedge z \right), \text{ distributivity} \\ &= \alpha(a) \odot \frac{1}{N} + \sum_{z \in \text{at}(B), z \neq q} \bar{\gamma}(a \wedge z), \text{ additivity of } \bar{\gamma} \\ &= \alpha(a) \odot \frac{1}{N} + \sum_{z \in \text{at}(B), z \neq q} \alpha(a) \odot \beta(z), \text{ definition of } \gamma, \bar{\gamma} \supseteq \gamma \supseteq \beta \\ &= \alpha(a) \odot \frac{1}{N} + (N-1) \left(\alpha(a) \odot \frac{1}{N} \right), \text{ definition of } \beta \\ &= N \left(\alpha(a) \odot \frac{1}{N} \right). \end{aligned}$$

We have shown:

$$\alpha(a) \odot \frac{1}{N} = \alpha(a) \cdot \frac{1}{N} \text{ for all } a \in A \text{ and } N = 1, 2, \dots \quad (3.8)$$

Fix now $k \in \{0, \dots, N\}$ and let $q_1, \dots, q_k$ be distinct atoms of B . Then

$$\begin{aligned}
\alpha(a) \odot \frac{k}{N} &= \alpha(a) \odot (\beta(q_1) + \cdots + \beta(q_k)), \text{ definition of } \beta \\
&= \alpha(a) \odot \beta(q_1 \vee \cdots \vee q_k), \text{ additivity of the consistent book } \beta \\
&= \bar{\gamma}(a \wedge (q_1 \vee \cdots \vee q_k)), \text{ definition of } \gamma \subseteq \bar{\gamma} \\
&= \bar{\gamma}((a \wedge q_1) \vee \cdots \vee (a \wedge q_k)), \text{ distributivity} \\
&= \bar{\gamma}(a \wedge q_1) + \cdots + \gamma(a \wedge q_k), \text{ additivity of } \bar{\gamma} \\
&= \alpha(a) \odot \beta(q_1) + \cdots + \alpha(a) \odot \beta(q_k), \text{ definition of } \gamma \\
&= k \left(\alpha(a) \odot \frac{1}{N} \right), \text{ definition of } \beta \\
&= k \left(\alpha(a) \cdot \frac{1}{N} \right), \text{ by (3.8)} \\
&= \alpha(a) \cdot \frac{k}{N}.
\end{aligned}$$

Generalizing (3.8), we have thus shown:

$$\alpha(a) \odot \frac{k}{N} = \alpha(a) \cdot \frac{k}{N}, \text{ for all } a \in A, \quad N = 1, 2, \dots \text{ and } k \in \{0, \dots, N\}. \quad (3.9)$$

Since the boolean algebras A and B have arbitrary numbers M, N of atoms respectively, recalling (3.6), the possible values of $\alpha(a)$ and k/N range over the totality of rational numbers r and s in the unit interval $[0, 1]$. From (3.9) we then have

$$r \odot s = r \cdot s \quad \text{for all rational numbers } r, s \in [0, 1].$$

Thus the $\odot$ operation agrees with the product operation $\cdot$ over a dense subset of $[0, 1]^2$. Since $\odot$ is assumed to be continuous, $\odot$ coincides with product over $[0, 1]^2$. $\square$

The continuity of $\odot$ is assumed to ensure that negligible errors/perturbations in the assessments $\alpha(x), \beta(y)$ of the events $x \in \text{dom}(\alpha), y \in \text{dom}(\beta)$ have a negligible effect on the values $\alpha(x) \odot \beta(y)$ assigned by the $\odot$ -book of α and β to the event $x \wedge y$. Dropping the hypothesis of continuity, still $\odot$ coincides with product on the dense set of all rational points in $[0, 1]^2$, which is amply sufficient for bets sold by real-life bookmakers.

3.3 Stone Duality Theorem

Our goal in this chapter is to extend Theorem 3.3 by proving that product does satisfy the conditions in the statement therein. For the proof in Theorem 3.8

we will use a number of fundamental results in measure theory [1] and boolean algebra theory [2]. The pace is faster than that of the previous chapters.

By a *maximal ideal* $\mathfrak{m}$ of a boolean algebra A we mean the kernel of a homomorphism $\eta \in \text{hom}(A)$, $\mathfrak{m} = \ker(\eta) = \{e \in A \mid \eta(e) = 0\}$. The *Stone space* of A is the set $\mu(A)$ of maximal ideals of A equipped with the following topology: a basic *clopen* (closed-and-open) of $\mu(A)$ has the form

$$X_a = \{\mathfrak{m} \in \mu(A) \mid a \in \mathfrak{m}\},$$

letting a range over all elements of A . By definition, an open set of $\mu(A)$ is an arbitrary union of clopens X_a . $\mu(A)$ is known as a *boolean space*, i.e., a compact Hausdorff space Y whose open sets are precisely the unions of clopens of Y . The topological product of two boolean spaces is easily seen to be a boolean space.

In Kolmogorov's probability theory, "events" form a σ -*algebra*, (i.e., a boolean algebra closed under countable unions) which is thought of by him as being more fundamental than the space Ω of sample points, alias, "elementary events". On page 61 of his paper "Complete Metric Boolean Algebras. Philosophical Studies", 77 (1995) 57–66. Kolmogorov writes:

...the notion of an elementary event is an artificial superstructure imposed on the concrete notion of an event. In reality, events are not composed of elementary events, but elementary events originate in the dismemberment of composite events.

For yes-no boolean events, the following result shows that the elementary event space is no more artificial than its dual event space. These two spaces are symbiotic like the two sides of a coin. In the final chapter of these handouts we will discuss an extension of this symbiosis to all compact Hausdorff spaces.

Theorem 3.4. (Stone representation theorem, see [2, 7.1-7.3] or any other book on boolean algebras) *For every boolean algebra A , the algebra $\text{clop}(\mu(A))$ of clopen sets of the Stone space $\mu(A)$, equipped with the distinguished elements $\emptyset$ and $\mu(A)$ along with the operations of complementation, union, and intersection, is isomorphic to A .*

Every boolean algebra A is isomorphic to the boolean algebra of clopen sets of some boolean space $Y \neq \emptyset$: $A \cong (\text{clop}(Y), \emptyset, Y, \neg, \cup, \cap)$.

While Proposition 1.5 identifies every finite boolean algebra with the powerset algebra of a finite set (with the discrete topology), Stone's theorem yields a much more general concrete representation of every, possibly infinite, boolean algebra as the algebra of clopens in a boolean space.

Generalizing Example 3.2, the following result states that if the boolean algebra C is generated by the union of two independent subalgebras A and B , then its Stone space is homeomorphic to the product of the Stone spaces of A and B , $\mu(C) \cong \mu(A) \times \mu(B)$:

Corollary 3.5. ([2, (1), p. 161], with a different but equivalent notation)
If A and B are independent subalgebras of C and the set $A \cup B$ generates C , then C is isomorphic to the boolean algebra $\text{clop}(\mu(A) \times \mu(B))$, where $\mu(A) \times \mu(B)$ is equipped with the product topology.

3.4 Probability space, Carathéodory Extension Theorem

The duality between events (=elements of a boolean algebra) and elementary events (=points in the Stone space) is further strengthened by Theorem 3.6. To this purpose, we prepare the following notation and terminology from elementary measure theory, [1].

Given a nonempty set Ω , by a σ -algebra we mean a subset of the power-set $\text{pow}(\Omega)$ containing Ω and closed under complementation and countable unions. Thus every σ -algebra is a very special kind of a boolean algebra of sets.

For X a nonempty subset of Ω we denote by $\sigma(X)$ the smallest σ -algebra in Ω containing X as a member.

A pair $(\Omega, \mathcal{A})$ consisting of a nonempty set Ω and a σ -algebra $\mathcal{A} \subseteq \text{pow}(\Omega)$ is called a *measurable space*.

Let $\mathcal{A}$ be a measurable space. A function $\mu: \mathcal{A} \rightarrow \mathbb{R} \cup \{\infty\}$ with $\mu(\emptyset) = 0$ is said to be *finite* if its range does not contain ∞ .

If there is a sequence $\Omega_1, \Omega_2, \dots$ of subsets of Ω such that $\Omega = \bigcup_n \Omega_n$ and $\mu(\Omega_n) < \infty$ for all n , we say that μ is σ -finite.

We say that μ is a *measure* on the measurable space $\mathcal{A}$ if μ has the following *σ -additivity property*: For any sequence of pairwise disjoint subsets Ω_i of Ω , $\mu(\bigcup_n \Omega_n) = \sum_n \mu(\Omega_n)$.

If, in addition, $\mu(\Omega) = 1$, we say that μ is a *probability measure*.

A triple $(\Omega, \mathcal{A}, \mu)$ is called a *measure* (resp., a *probability*) *space* if $(\Omega, \mathcal{A})$ is a measurable space and μ is a measure (resp., a probability measure) on $\mathcal{A}$. We say that $(\Omega, \mathcal{A}, \mu)$ is σ -finite if so is μ .

In particular, when Ω is a topological space and $\mathcal{O}$ is the family of open sets of Ω , then the σ -algebra $\sigma(\mathcal{O})$ is called the *Borel σ -algebra* on Ω , denoted $\mathcal{B}(\Omega)$. Each element of $\mathcal{B}(\Omega)$ is called a *Borel set*. Every measure μ on $\mathcal{B}(\Omega)$ is said to be a *Borel measure*. The pair $(\Omega, \mathcal{B}(\Omega))$ is called a *Borel measurable space*. The triple $(\Omega, \mathcal{B}(\Omega), \mu)$ is said to be a *Borel measure space*.

For any two Borel measurable spaces $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$ we denote by $\mathcal{B}(\Omega_1 \times \Omega_2)$ the σ -algebra of subsets of $\Omega_1 \times \Omega_2$ generated by all sets of the form $X \times \Omega_2$ and $\Omega_1 \times Y$, where $X \in \mathcal{B}(\Omega_1)$ and $Y \in \mathcal{B}(\Omega_2)$. $\mathcal{B}(\Omega_1 \times \Omega_2)$ is a Borel measurable space.

Theorem 3.6. (Carathéodory extension theorem, weak form. See [1, Theorem 1.41, p.18]).² *Let $X \neq \emptyset$ be a boolean space. Then every state of the boolean algebra $\text{clop}(X)$ (uniquely) extends to a probability measure on the σ -algebra $\mathcal{B}(X)$ of Borel sets of X .*

In its full form, Theorem 3.6 states that the state space of the boolean algebra $\text{clop}(X)$ is *affinely homeomorphic* ($\cong$) to the space $\mathbf{m}(X)$ of regular Borel probability measures on X equipped with the weak topology. For every boolean algebra A , letting $S(A)$ be the state space of A endowed with the restriction of the product topology of $[0, 1]^A$, Corollary 2.8 and Theorem 3.4 then yield:

$$\text{the set of finitely consistent maps of } A = S(A) \cong \mathbf{m}(\mu(A)). \quad (3.10)$$

Theorem 3.7. (Weak form of the product measure theorem. See [1, Theorem 14.14, p.277], or any other textbook on measure theory). *For each $i = 1, 2$, let $(\Omega_i, \mathcal{B}(\Omega_i), \mu_i)$ be σ -finite Borel measure spaces. Then there is a σ -finite measure χ on the σ -algebra $\mathcal{B}(\Omega_1 \times \Omega_2)$ satisfying $\chi(X \times Y) = \mu_1(X) \cdot \mu_2(Y)$ for all $X \in \mathcal{B}(\Omega_1)$ and $Y \in \mathcal{B}(\Omega_2)$.*

3.5 De Finetti's Consistency implies the Product Law

We are now ready to derive from de Finetti's consistency theorem the “product law” of the standard definition of stochastically independent events. In fact, the following theorem shows that a continuous operation $\odot: [0, 1]^2 \rightarrow [0, 1]$ preserves the consistency of *arbitrary* books on algebraically independent (possibly infinite) boolean algebras iff $\odot$ coincides with product.

Theorem 3.8. *Product is the only continuous $[0, 1]$ -valued operation $\odot$ defined on $[0, 1]^2$ having the following property:*

For any consistent books $\alpha: \text{dom}(\alpha) \rightarrow [0, 1]$ and $\beta: \text{dom}(\beta) \rightarrow [0, 1]$ on finite subsets $\text{dom}(\alpha) \subseteq A$ and $\text{dom}(\beta) \subseteq B$ of (possibly infinite) independent subalgebras A and B of a boolean algebra C , letting the finite subset $C_\wedge$ of C be defined by $C_\wedge = \{q \in C \mid q = a \wedge b \text{ for some } a \in \text{dom}(\alpha) \text{ and } b \in \text{dom}(\beta)\}$ it follows that:

- (i) *Whenever $a, a' \in \text{dom}(\alpha)$ and $b, b' \in \text{dom}(\beta)$ satisfy $a \wedge b = a' \wedge b'$, then $\alpha(a) \odot \beta(b) = \alpha(a') \odot \beta(b')$; in other words, $\odot$ satisfies condition (3.2), which ensures the existence of the product book γ on $C_\wedge$ by the stipulation*

$$\gamma: a \wedge b \in C_\wedge \mapsto \alpha(a) \cdot \beta(b) \text{ for all } a \in \text{dom}(\alpha) \text{ and } b \in \text{dom}(\beta). \quad (3.11)$$

- (ii) *γ is a consistent book on $C_\wedge$.*

² Also see Corollary 1 in T.Tao, <https://terrytao.wordpress.com/2009/01/03/>.

Proof. In Theorem 3.3 it is proved that *at most one* continuous operation $\odot: [0, 1]^2 \rightarrow [0, 1]$ has properties (i)-(ii), and if such operation exists it coincides with product. There remains to be proved that product has both properties.

(i) By Stone's theorem 3.4 we may argue under the assumption

$$A = \text{cllop}(X) \quad \text{and} \quad B = \text{cllop}(Y), \quad \text{for some boolean spaces } X \text{ and } Y.$$

Since A and B are independent boolean subalgebras of C , they are also independent subalgebras of the boolean algebra $C_{A \cup B}$ of C generated by the set $A \cup B$. Without loss of generality, we may henceforth assume $C = C_{A \cup B}$. By Corollary 3.5, we may argue assuming the identities $C = C_{A \cup B} = \text{cllop}(X \times Y)$, where the boolean space $X \times Y$ is the cartesian product of X and Y equipped with the product topology.

The $(\Rightarrow)$ -direction of de Finetti's consistency theorem 2.7 yields states $\bar{\alpha}$ of A and $\bar{\beta}$ of B , respectively extending the consistent books α and β . Carathéodory extension theorem 3.6 yields Borel probability measures ϕ and ψ over the Borel σ -algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ generated by the clopen sets of the spaces X and Y , respectively extending $\bar{\alpha}$ and $\bar{\beta}$:

$$\bar{\alpha}(a) = \phi(a) \quad \text{for all } a \in \text{cllop}(X) \quad \text{and} \quad \bar{\beta}(b) = \psi(b) \quad \text{for all } b \in \text{cllop}(Y).$$

An application of Theorem 3.7 to the probability spaces $(X, \mathcal{B}(X), \phi)$ and $(Y, \mathcal{B}(Y), \psi)$ yields a (probability) measure χ on $\mathcal{B}(X \times Y)$ satisfying

$$\chi(U \times V) = \phi(U) \cdot \psi(V) \quad \text{for all } U \in \mathcal{B}(X) \text{ and } V \in \mathcal{B}(Y).$$

Let the function $\bar{\gamma}: C_{A \cup B} \rightarrow [0, 1]$ be the restriction of χ to $C_{A \cup B}$. Since $\bar{\gamma}(c) = \chi(c)$ for all $c \in C_{A \cup B}$ then $\bar{\gamma}$ is a state of the boolean algebra $C_{A \cup B} = \text{cllop}(X \times Y)$. Furthermore, for all $a, a' \in A$ and $b, b' \in B$,

$$\text{if } a \wedge b = a' \wedge b' \text{ then } \bar{\gamma}(a \wedge b) = \bar{\alpha}(a) \cdot \bar{\beta}(b) = \bar{\alpha}(a') \cdot \bar{\beta}(b').$$

This amounts to saying that the books α and β satisfy condition (3.2). Let $\gamma: C_{\wedge} \rightarrow [0, 1]$ be the restriction of $\bar{\gamma}$ to $C_{\wedge}$. Since the states $\bar{\alpha}$ and $\bar{\beta}$ respectively extend the books α and β , we conclude that γ is the desired product-book on $C_{\wedge}$ satisfying condition (3.11).

(ii) This follows from the $(\Leftarrow)$ -direction of de Finetti's consistency theorem, because the book γ constructed in (i) is the restriction to $C_{\wedge}$ of a state of the boolean algebra C . $\square$

As a companion of (2.18) we may now write:

$$\boxed{\text{consistency} + \text{independence}} \Rightarrow \text{product law} \quad (3.12)$$

References

1. A. Klenke, Probability Theory, A Comprehensive Course, Second Edition, Universitext, Springer, Berlin, 2013.
2. S. Koppelberg, General Theory of Boolean Algebras. In: Handbook of Boolean Algebras (Vol.1). Edited by J.D.Monk with the cooperation of R. Bonnet, Elsevier, Amsterdam, 1989.

Recent Developments

As we have seen in Chapter 2, de Finetti's consistency theorem implies the additivity axiom for disjunctions of incompatible events. Likewise, in Chapter 3 we have shown that de Finetti's consistency theorem implies the product law for stochastically independent events.

What remains to be done?

MV-algebras and their maximal spectral spaces

While de Finetti championed finite additivity, Kolmogorov introduced the countable additivity axiom for his probability spaces $(\Omega, \mathcal{F}, \mu)$ as follows:

We limit ourselves, arbitrarily, to only those models which satisfy Axiom VI. This limitation has been found expedient in researches of the most diverse sort.

Kolmogorov, A.N. "Grundbegriffe der Wahrscheinlichkeitsrechnung". Springer, Berlin, 1933. English translation in: "The Foundations of the Theory of Probability", Chelsea Publishing Company, New York, 1950. See p.15 in the English translation.

Axiom VI is a continuity axiom which is immediately seen to be equivalent to countable or σ -additivity. Thus "events" are understood as elements of a σ -algebra $\mathcal{F}$ of subsets of the set Ω of "elementary events". μ is a probability measure on $\mathcal{F}$.

Can the (arbitrariness of) Kolmogorov's σ -additivity axiom be reconciled with finitely additive probability based on de Finetti's logic-operational consistency notion?

A first partial positive answer is given by Carathéodory's extension theorem 3.6, which shows that the states of any boolean algebra A can be identified, via an affine homeomorphism, with the regular Borel probability measures on the Stone space of A . The latter turns out to be the most general

boolean space. However, boolean spaces are a too restricted class for the demands of modern probability theory.

A more satisfactory answer for all compact Hausdorff spaces is provided if one replaces boolean algebras (resp., boolean logic) by MV-algebras (resp., Łukasiewicz logic).

Il n'y a pas de doute possible, depuis les belles recherches de MM. Łukasiewicz, Reichenbach, Mazurkiewicz, etc., que le calcul des probabilités peut être considéré comme une logique à plusieurs valeurs (précisément: à une échelle continue de valeurs), et que ce point de vue est le plus propre à éclaircir dans les fondements la notion et la logique de la probabilité. Mais ce but est loin d'être atteint par la seule conclusion, de nature purement formelle, que le calcul des probabilités est une logique à plusieurs valeurs: cette conclusion n'est utile que comme point de départ, elle ne constitue pas une façon de résoudre le problème, mais seulement une façon heureuse de le poser nettement. Je crois donc utile d'examiner, comparer et discuter les diverses interprétations qu'on peut donner de cette même théorie des probabilités conçue comme logique à plusieurs valeurs.

De Finetti, Introduction of "La logique de la probabilité". In: Actes du congrès international de philosophie scientifique, Sorbonne, Paris 1935, Hermann et C^{ie} Éditeurs, pp. 565–573, 1936.

MV-algebras stand to Łukasiewicz infinite-valued logic as boolean algebras stand to boolean logic. They are defined by the following equations:

$$\begin{aligned} x \oplus (y \oplus z) &= (x \oplus y) \oplus z \\ x \oplus y &= y \oplus x \\ x \oplus 0 &= x \\ \neg\neg x &= x \\ x \oplus \neg 0 &= \neg 0 \\ \neg(\neg x \oplus y) \oplus y &= \neg(\neg y \oplus x) \oplus x. \end{aligned}$$

Let 1 be an abbreviation of $\neg 0$. Adding the idempotence equation $x \oplus x = x$, we obtain the defining equations of boolean algebras, where $\vee = \oplus$ and $x \wedge y = \neg(\neg x \vee \neg y)$.¹

With the derived conjunction operation $x \odot y = \neg(\neg x \oplus \neg y)$, a state of an MV-algebra A is defined precisely as in the boolean case, as a function $\sigma: A \rightarrow [0, 1]$ satisfying the normality condition $\sigma(1) = 1$ and the additivity condition $\sigma(a \oplus b) = \sigma(a) + \sigma(b)$ whenever $a \odot b = 0$.

A *maximal ideal* $\mathfrak{m}$ in an MV-algebra A is the kernel of a homomorphism of A into the *standard MV-algebra* $[0, 1]$ equipped with the operations $\neg x = 1 - x$

¹ For a comprehensive account see R. Cignoli, I. M. L. D'Ottaviano, D. Mundici, "Algebraic Foundations of many-valued Reasoning", Trends in Logic, Vol.7, Kluwer, Dordrecht, 2000. Reprinted, Springer Science & Business Media, (2013).

and $x \oplus y = \min(1, x + y)$. The *maximal spectral space* $\mu(A)$ of any MV-algebra A is defined by analogy with the Stone space of a boolean algebra, by stipulating that a basic closed set of $\mu(A)$ has the form

$$X_a = \{\mathfrak{m} \in \mu(A) \mid a \in \mathfrak{m}\},$$

letting a range over all elements of A . We have the following fundamental result:

$\mu(A)$ is the most general possible compact Hausdorff space.

Finite vs. countable additivity: the Kroupa-Panti theorem

The generalization of the Carathéodory extension theorem 3.6 to every MV-algebra A is given by the Kroupa-Panti theorem.² In its full form, the theorem states:

For any MV-algebra A there is an affine homeomorphism κ_A between the state space of A equipped with the restriction of the product topology of the compact cube $[0, 1]^A$, and the weakly compact space of regular Borel probability measures on the maximal spectral space $\mu(A)$.

As a consequence, in the present author's paper "Rota's Fubini lectures: The first problem", *Advances in Applied Mathematics*, 125 (2021) 102153, the following theorem is proved:

For every Kolmogorov probability space $(\Omega, \mathcal{B}(\Omega), \mu)$, with $\mathcal{B}(\Omega)$ the σ -algebra of Borel sets of a compact Hausdorff space Ω , and μ a regular probability measure on $\mathcal{B}(\Omega)$, there is an MV-algebra A and a state σ of A such that

$$(\Omega, \mathcal{B}(\Omega), \mu) \cong (\mu(A), \mathcal{B}(\mu(A)), \kappa_A(\sigma)). \quad (4.1)$$

Therefore, Kolmogorov triples $(\Omega, \mathcal{B}(\Omega), \mu)$ with μ a (countably additive) regular probability measure on the Borel sets of a compact Hausdorff space Ω are mathematically equivalent to pairs (A, σ) with A = the MV-algebra of all continuous $[0, 1]$ -valued functions on Ω , and σ a (finitely additive) state of A .

The identifications in (3.10) can be thus extended to every MV-algebra A , since also *de Finetti's consistency theorem holds for A* . See Theorem 3.2 in the paper "De Finetti theorem and Borel states in $[0, 1]$ -valued algebraic logic", *International Journal of Approximate Reasoning*, 46 (2007) 605-616, by J. Kühr and the present author. The latter showed that *A also satisfies the MV-algebraic counterpart of de Finetti's Fundamental Theorem of Probability*

² T. Kroupa, "Every state on a semisimple MV-algebra is integral", *Fuzzy Sets and Systems*, 157 (2006) 2771-2782. G. Panti, "Invariant measures in free MV-algebras", *Communications in Algebra*, 36 (2009) 2849-2861.

1.12, and Theorem 3.8 on the product law for independent events. See, respectively, Theorem 6.1 in the author's cited paper on Rota's Fubini lectures, and Theorem 2.3 in the article "Betting on continuous independent events," *Soft Computing*, 23 (2019) 2289–2295.

An interesting problem is to devise a class of logic-algebraic structures capable of providing the extension of the correspondence (4.1) to all Kolmogorov probability spaces over locally compact spaces.

It is a consistent bet that de Finetti's consistency notion and his theorems will play a key role in solving these problems.

Concluding Remarks

Like precious pearls, de Finetti's theorems in these handouts can be drawn from the depths of the original sources. Readers who want to immerse themselves in these sources will find pages of brilliant paraphrases, suggestive analogies, rhetorical figures, delicate nuances, in great abundance compared to the definitions, lemmas, corollaries, proofs in these handouts. They will find fewer traces of boolean algebras, their homomorphisms, and their states. But to the attentive reader, de Finetti's theorems will shine with their own light regardless of the concise algebraic presentation given here.

Leaving aside the unfortunate translation "coherence" instead of "consistency" of the Italian noun "coerenza", de Finetti's writing style is notoriously difficult to understand, even for mathematicians who know the Italian language. The number of English-speaking mathematicians who have really tried to interpret what he has written is very small—a fate that no mathematician would wish to wish her/his theorems.

The Fundamental Theorem of Asset Pricing is just de Finetti's consistency-theorem-*cum*-discounting. But de Finetti is not usually given credit in the finance literature for his discovery of this result four decades earlier. His consistency theorem is also absent in most mathematical books on probability. Several proofs found in articles published on mathematical journals have the problems mentioned in the Preface and in Remarks 2.9(i)-(iii). De Finetti's exchangeability theorem has long been regarded by probabilists as an isolated curiosity. Well-known books and articles dealing with Boole's probabilistic inference problem do not mention what de Finetti named his "Fundamental Theorem of Probability".

At the end of the day, the reader may wonder what is the *meaning* of the numbers ρ_i , ($i = 1, \dots, m$) assigned to the events h_i lying in the boolean algebra A in de Finetti's consistency theorem. Do these numbers stand for lottery ticket prices, bookmakers' odds, risk neutral probabilities, marginal utility ratios, cardinal utilities? Or, do they represent something even deeper?

Given the nature of these handouts, which is specifically aimed at undergraduate students in mathematics, the above problems, have not been considered. In these final pages we have only hinted at the infinite-valued

extension of the semantics of the Łukasiewicz formulas ϕ_i that code continuous random variables h_i rather than boolean yes-no events. Far beyond the possible outcomes of football matches or horse racing, these formulas provide efficient coding of random variables of interest in various areas of uncertainty management, from medical practice to finance.

Last, but not least, pairs (A, σ) with A an MV-algebra and σ a state of A , i.e., a finitely consistent function $\sigma: A \rightarrow [0, 1]$, extend the scope of de Finetti's consistency theorem to Kolmogorov's $(\Omega, \mathcal{F}, \mu)$ triples on all compact Hausdorff spaces Ω . An affine homeomorphism reconciles de Finetti's finitely additive probability and Kolmogorov's axiom of countable additivity.

If, as de Finetti strongly argued, probability does not exist, mathematics does. And in these handouts, elementary algebra, analysis, geometry, topology, logic, combinatorics and measure theory have worked in synergy to offer our young readers some of his far-reaching theorems.

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