Inconceivability in Kant, Frege, and the Tractatus

WORDCOUNT: 23533

Abstract

Frege intended to prove that all truth of arithmetic could be grounded on our knowledge about logic. This raises a question as to how he intended to ground our knowledge about logic. Dummett, Hacker, and De Pierris maintain that he appealed to self-evidence at this juncture. Stroud argues that he left this problem unsolved. Conant argues that he gave the same answer that Kant and Wittgenstein gave: the negation of basic laws is impossible to grasp. I agree with Conant that Kant, Frege, and Wittgenstein held similar epistemological views, but not the view that Conant has in mind. Rather, they thought that the principles at the basis of their systems are ideally impossible to reject.

1. The Secondary Problem of Pure Reason

Kant, Frege, and Wittgenstein thought that the truths of arithmetic can be known a priori. One can be justified in believing that 7+5=12 independently of experience. This raises a question about the source of this justification, which is part of what Kant in the first critique calls "the general problem of pure reason" (B19).[1] Kant appealed to our knowledge about pure intuitions to account for our knowledge of the truths of arithmetic (B16). Frege and Wittgenstein, by contrast, thought that our knowledge of logic and language would suffice (Frege 1884*,* §12, §89 and TLP*,* 6.2).

This narrative seems to raise an additional question: how did Kant, Frege, and Wittgenstein account for our knowledge of pure intuitions and logic? Imagine that Frege's basic laws could give a foundation of arithmetic as he envisioned; this would still leave open the question: how are we justified in believing the basic laws themselves? And this seems to apply, quite generally, to any theory reducing arithmetical knowledge to a second type of knowledge. In this paper, I want to focus on this problem, which I label *the secondary problem of pure reason*. My suggestion is that Kant, Frege, and Wittgenstein offered a similar solution at this juncture: the basic principles of their theories are justified because denying them would be ideally impossible. Wittgenstein, is most straightforward on this matter:

[1] As customary, quotations from the first critique are given with 'A' and 'B' for first and second edition. I use Guyer and Wood's translation throughout, although I mention an issue that Hanna (2022) raises with how Guyer and Wood translate '*glauben'* in the preface to the second edition, see footnote 29.

What makes logic a priori is the *impossibility* of illogical thought. (TLP, 5.4731)[2]

This should certainly give us pause. Wittgenstein's solution, prima facie, seems both wrong and unsatisfactory. It seems wrong because, at least prima facie, people *can* reason illogically. Anyone who spent some time studying logic knows how easy it is to make logical mistakes. But Wittgenstein's solution suggests that there cannot be logical mistakes.[3] It seems unsatisfactory because it is unclear how one could be justified in believing something just because one cannot think its negation. A subject could be unable, for whatever reason, to think that π approximates to 3.14, but that would hardly justify the subject in believing that π does not approximate to 3.14.[4]

I should not say much about this now, since much of this paper discusses just how this solution can be made to look neither wrong nor unsatisfactory; or, better, I discuss why Kant, Frege, and Wittgenstein thought that this was neither wrong nor unsatisfactory. To lay down my cards, I should only anticipate that the impossibility that Wittgenstein is concerned with is not a psychological limitation. It has nothing to do with a given subject not being able to think something. According to Wittgenstein illogical thought is impossible *in general* and somehow this justifies our logical thinking. Conant (1992) argued that the

[2] I use the Pears and McGuinness translation throughout, I refer to the German when this might cause confusion.

[3] Indeed, he explicitly commits to this idea "Logic must look after itself […] In a certain sense, we cannot make mistakes in logic" (TLP, 5.473). See also NB, 2.

[4] A third important problem is this: in a sense, Wittgenstein's logical aliens are now commonplace; thus, they are hardly impossible. For instance, Priest (1987) believes in true contradictions. I postpone to another time a discussion about this.

impossibility to think illogically amounts to an impossibility to *grasp* a thought that violates the laws of logic, because there are no such thoughts. I argue that the impossibility at play here amounts to an impossibility to *assert* illogical thoughts (contradictions). But let us proceed in order.

Wittgenstein contrasts his own solution to what he takes to be Frege's and Russell's. In dictation, and in the *Tractatus*, he says that Frege and Russell appealed to self-evidence, thus giving a psychologistic solution to the secondary problem of pure reason.

> One of the great difficulties which is in logic was this—what is the criterion for a proposition being a proposition of logic.
>
> By some it was taken to be self-evidence. This is what both Russell and Frege said. At certain points where this is a difficulty one tends to bring in psychology.
>
> The word self-evident is taken from psychology.
>
> It is what everybody in his right senses would say is true.
>
> It is the highest degree of plausibility. But it is clear that what is self-evident [to somebody] may not be to another.[5] (WSM, 270)

Never mind whether this is a correct assessment of how Russell meant to solve the secondary problem of pure reason. According to some interpreters, this is a faithful depiction of Frege's approach. For instance, De Pierris (1988) argues that Frege appealed to self-evidence.[6] Laws of logic "neither need nor admit of proof" (Frege 1884*,* 4). According to

[5] See also TLP, 6.1271 5.4731.

[6] See also Hacker 1986, 43, among others.

De Pierris: "Frege does not attempt to explain what guarantees that our laws of logic are the laws of truth [he] treats [logic] as a piece of accepted knowledge" (1988, 310). Thus, all evidence we can give to support them is that they feel, as Russell puts it, "luminously evident" (Russell 1912, 64).[7]

One problem with this interpretation is that it suggests that Frege, the champion of anti-psychologism, placed a psychological element at the core of his system. Wittgenstein takes this to be a reductio, while other interpreters try to remove the psychological element in Frege's alleged appeal to self-evidence. For instance, Dummett (1980) argues that Frege's self-evidence is a non-psychological acquaintance with platonic entities, analogous to sense perception. And Stroud (2018) argues that Frege is not truly appealing to a technical

[7] I should anticipate that at this juncture interpretative issues overlap with translation issues. De Pierris translates '*einleuchten'* as 'self-evident,' just like Geach does (Frege 1960, 234). Ebert and Rossberg translate it as 'obvious' in the same passage (Frege 2016, 253) because, they say: "Frege uses the term to indicate that something, such as a proof, is easily understood or accepted" (Ebert and Rossberg 2016, xxii). In TLP 6.1271, where Wittgenstein criticizes Frege's alleged appeal to self-evidence, he uses the term '*einleuchten.'* This might suggest that he was targeting the notion that De Pierris and Geach translate as 'self-evident.' But then it is surprising that Frege does not use it in the passages usually quoted in connection with the secondary problem of pure reason (Frege 1884*,* §14; 1893, xvii). Nor does he use 'selbstverstandlich,' the word that Ebert and Rossberg too translate as 'self-evident.' I maintain that translation issues can be set aside: regardless of translation, if Frege intended to claim that the basic laws are self-evident, he would have used at least one of the words (*einleuchten* or selbstverstandlich) in the crucial passages.

epistemological notion of self-evidence, but rather he is leaving the secondary problem of pure reason unsolved:

> "The question why and with what right we acknowledge a law of logic to be true," Frege says, "logic can answer only by reducing it to another law of logic." That leaves the 'ground' or justification of those laws themselves so far unexplained. It says nothing to suggest that there even could be such an explanation. (Stroud 2018, 210 quoting Frege 1893, xvii)

I call 'the controversial passage' the paragraph in the *Grundgesetze* that Stroud is quoting here (Frege 1893, xvii). This is quoted by many interpreters, in full or in parts, when discussing Frege's views on the epistemology of logic.[8] Now, Stroud's reading of the controversial passage might remove the psychological element from Frege's alleged appeal to self-evidence, but it does so at a great cost.

First, Stroud's interpretation breaks down Frege's analogy between logical laws and scientific laws. Macfarlane (2002) argued that Frege's laws of logic are *weakly normative*.[9] According to this interpretation, the laws of logic "are descriptive in content but have prescriptive consequences" (ibid. 37). Stroud seems to endorse this reading, as I do, but his interpretation of the controversial passage is in tension with it. Much of the strength of this weakly normativist interpretation is that it makes laws of logic analogous to scientific laws,

[8] See also Dummett 1980, 510; Hacker 1986, 43; De Pierris 1988, 305-6; Conant 1992, 78.

[9] This idea is also in Friedman 1985, although Friedman uses a different terminology. Friedman says that for Frege the laws of logic are analogous to scientific laws, the only difference being that they are *maximally general* (1985, endnote 72).

the only difference being their generality. But the analogy breaks down if we add that *unlike scientific laws*, the laws of logic cannot be justified.

Second, Stroud's view seems to leave a gap in Frege's theory. Our knowledge of arithmetic is justified by our knowledge of the laws of logic. But our knowledge of the laws of logic is unaccounted for. Laws of logic are either "a basic part of our repertoire or […] just part of the way things are" (Stroud 2018, 217), but we seem to have no way of knowing that for certain. Thus, in this view the ultimate ground of our arithmetical knowledge is somewhat unjustified. Stroud suggests that there is nothing problematic with leaving the secondary problem of pure reason unsolved; it is merely a psychological fact that we tend to find this unsatisfactory. However, he gives no reason to think that Frege endorsed this view. Indeed, Stroud presents this as his own attitude. Accordingly, it is reasonable to have a sceptical attitude here, especially given what Frege writes concerning the epistemic status of axioms.

> We cannot accept a thought as an axiom if we are in doubt about its truth. (Frege 1914, 205)

Similar remarks can be made about Dummett's (1980) reading. If the justification of laws of logic derives from direct acquaintance with platonic entities, then the laws of logic are different from scientific laws in the source of their justification. But Frege seems to think that the only difference should be that laws of logic are more general than any other law (Frege 1893, xv). Furthermore, Dummett's view also seems to leave a gap in Frege's theory, since the idea that we can be directly acquainted with platonic entities is somewhat unqualified. As De Pierris puts it "the view does not seem to have more content than this metaphor or simile with the direct acquaintance with physical objects by means of sense perception" (1988, 297).

These difficulties give us reason, at least prima facie, to search an alternative reading. Conant suggests that, for Frege, the laws of logic are "constitutive of the possibility of thought" (1992, 57). While Macfarlane and Stroud maintain that no *unmistaken* thought can violate Frege's laws of logic, Conant maintains that no thought *at all* can violate them and this somehow justifies them. In this interpretation, Frege's solution is the same as Wittgenstein's.

This reading removes the gap in Frege's theory. There is an explanation as to how we can be justified to believe the laws of thought. And this explanation is somewhat analogous to the explanation we may give about natural laws. The laws of motion, for instance, are laws that no possible moving body can violate, and scientists inspect moving bodies to determine what laws of motion there are. Analogously, the laws of thought are laws that no possible thought can violate, and logicians inspect thoughts to determine what laws of thought there are. The only difference is that laws of thought are more general than laws of motion.

This interpretation creates a tension between the justification of the laws of logic and other aspects of Frege's theory, namely semantic compositionality, and the idea that the laws of logic have a sense.[10] Frege, according to Conant, claims that violations of a law of logic are not thoughts at all. For instance, there is no thought corresponding to the formula ¬(P->P), this is just nonsense. Now, by semantic compositionality, the negation of nonsense is also nonsense. Therefore, ¬¬(P->P) is nonsense. Beyond a doubt Frege accepted double negation

[10] The idea of a tension between the impossibility of illogical thought and the conception of logic as the maximally general science is also discussed by Ricketts (1985).

elimination, therefore this commits him to the idea that P->P is nonsense. But this is an instance of basic law I!

It cannot be overestimated how great a cost this is for Frege. Frege maintains that the sense of a sentence is the way in which it displays its reference, and there are only two possible references of any sentence, namely the True and the False (Frege, 1982). In this framework, the view that basic laws are nonsense entails that they fail to display the True. Therefore, the ground on which Frege meant to give a foundation of arithmetic crumbles under his feet.

I should emphasize that Conant is aware that his reading creates a tension in Frege's theory. He maintains that Frege did not realize that his views on the epistemology of logic were ultimately self-defeating.[11] Conant describes this as an intermediary step on the road to Wittgenstein's more perspicuous theory of arithmetic. According to Conant, Wittgenstein can avoid this problem since he rejects Frege's idea that laws of logic have a sense.[12]

[11] Notably, Conant and Diamond (2004) argue, contra Williams (2004), that TLP is in no sense contradictory or self-defeating. Therefore, in Conant's reading, Wittgenstein is overall less likely to have a self-defeating lapsus than Frege. Here I do not argue that TLP contains lapsus, except for a brief remark in footnote 70. But, in my interpretation, it is overall less likely that Frege would have a lapsus than Wittgenstein.

[12] Of course, this depends on how one reads the TLP distinction between nonsense [*unsinn*] and senseless [*sinnlos*]. Here I am presupposing Conant's reading of the distinction for the sake of argument.

> Propositions show what they say: tautologies and contradictions show that they say nothing […] Tautologies and contradictions lack a sense (TLP, 4.461)

Conant presents this as a return to Kant's solution of the secondary problem of pure reason. That is, Kant, like Frege, claims that what makes logic a priori is the impossibility of illogical thought. But, unlike Frege, Kant embraces the idea that the laws of logic say nothing.

> Logic contains the absolutely necessary rules of thought without which there can be no employment whatsoever of the understanding. (A52/B76)

This echoes TLP 5.4731. If Kant means by 'employment of the understanding' something close to what Wittgenstein means by 'thought' then here he is saying that there can be no illogical thought.

I see three immediate problems with Conant's reading. First, Kant was not a logicist. He thought that our knowledge of arithmetic is grounded on our knowledge about pure intuitions of space and time. I agree with Conant that for Kant there can be no illogical thought, and logic is somewhat contentless; this reflects Kant's syllogistic conception of logic. However, for Kant, our knowledge of mathematics rests on our substantive knowledge about intuitions. The precursor of Frege's solution to the secondary problem of pure reason are Kant's views about intuition, not his views about logic. Indeed, scholarly works on Kant's philosophy of mathematics tend to devote a great deal of attention to his theory of intuition (e.g. Hintikka 1967; Kitcher 1975; Friedman 1985; Aliston 1988; Brittan 1992; Sutherland 2004). By contrast, Conant focuses on Kant's views about logic, just like others who studied connections between Kant and the founders of analytic philosophy (e.g. De Pierris 1988; MacFarlane 2002; Stroud 2018).

Second, Conant's reading makes Frege defeat himself at a juncture where he was very unlikely to make a mistake. I have no principled reason to exclude that Frege might have committed to views that entail a contradiction, as he did. But, as I shall argue, Conant's view entails that Frege confused the notions of *grasping* and *asserting*. Now, each of Frege's technical works begins by drawing the distinction between the judgement stroke and the horizontal or content stroke (Frege 1879 §2; 1893, §5).[13] This distinction corresponds to the grasping-asserting distinction. It seems very unlikely that Frege would confuse these very notions. Thus, embracing Conant's reading should be considered a last resort.

Third, Frege and Wittgenstein held different theories of meaning. Crucially, Wittgenstein recognizes two notions (nonsense [*unsinn*] and senseless [*sinloss*]) that may correspond to Frege's 'nonsensical string of symbols.' In TLP a proposition has a sense only if it partitions the space of possibilities. For Frege a proposition has a sense only if it is a name of the True, or the False. When Wittgenstein says that tautologies are senseless (TLP 4.461) he is emphatically not saying that they are nonsense *in Frege's sense*. For them to be nonsense in Frege's sense, tautologies would have to be neither true not false. But tautologies are unconditionally true (ibid.). Conant's comparative study does not seem to take this terminological discrepancy into account.

These problems give us, at least prima facie, reason to investigate further Kant's, Frege's and Wittgenstein's solution to the secondary problem of pure reason. This paper develops a new interpretation. The central suggestion is that Kant, Frege, and Wittgenstein thought that certain principles are ideally impossible to *reject*. In Kant's case, a thought about

[13] See also, among others, Frege 1906c, 184 and 1906d, 185.

something that violates the principles of pure intuition can be grasped. As Kant puts it: "I can think whatever I like, as long as I do not contradict myself" (Footnote Bxxvi, See also JL 14-15). But if one truly understand the principle, say, that "between two points only one straight line is possible" (B202-7), then one cannot deny it. Analogously, for Wittgenstein one can grasp a contradiction by visualizing its truth-table. But if one does grasp a contradiction, then one cannot seriously reject it.

The rest of this paper has the following structure. In section 2 I discuss Kant's solution to the secondary problem of pure reason. I briefly discuss Kant's theory of intuition to argue that it is indeed knowledge about intuitions that we should focus on at this juncture. Then I draw a comparison between Kant's and Frege's terminologies to argue that, in his own terms, Kant held that the axioms of intuition are ideally impossible to reject. In section 3, I focus on the controversial passage. First, I discuss the self-evidence interpretation. This is the view that Frege appealed to self-evidence, or that he refused to answer the secondary problem of pure reason. Then I discuss the grasping interpretation. This is the view that, for Frege, the negation of basic laws cannot be grasped. I argue that neither interpretation stands scrutiny, and I propose my own. In section 4, I draw a comparison between Wittgenstein's and Frege's terminologies to argue that, in his own terms, Wittgenstein held that tautologies are ideally impossible to reject. Conclusions follow suit.

2. Kant's Solution

The secondary problem of pure reason arises when one accounts for a certain type of knowledge (primary target) in terms of a second type of knowledge (secondary target), it calls for a justification of the secondary target. Consequently, this problem can take different forms even when the primary target is the same. Kant, Frege, and Wittgenstein

have a common primary target (arithmetical knowledge), but they have different secondary targets; Kant, unlike Frege and Wittgenstein, appealed to intuition. A discussion about their solutions to the secondary problem of pure reason needs to take this into account.

In Kant's case, this approach requires that, before discussing his solution to the secondary problem of pure reason, I outline his theory of intuition, and how this theory relates to his philosophy of logic. Having done this, I give a definition of *inconceivability* and I suggest that this notion plays a crucial role in Kant's solution. This definition comes in handy later, in sections 3 and 4, where I argue that Frege's and Wittgenstein's solutions to their version of this problem are analogous to Kant's.

Pure intuition, empirical intuition, and understanding

The German word *'anschauung'* is customarily translated as '*intuition'* in the first critique. There are very good reasons for this, chief among which is that Kant himself translated the word in Latin as '*intuitus'* (e.g. B72, and A320/B376–7), yet this practice is somewhat infelicitous. An intuition in ordinary English is a thought that one is inclined to consider plausible based on instinctive feelings, as opposed to rational reasoning. Contemporary philosophers tend to use this word with a meaning closely related to the ordinary one. Intuitions are typically a type of knowledge, a belief, a disposition to believe, or an "intellectual seeming" (Bealer, 1998). This is not what Kant meant by *anschauung*.[14]

[14] This point is well-rehearsed, Carus (1892) suggests translating *anschauung* as 'at-sight.' And, more recently, Hatfield (2004) remarked that the word's etymological sense is "looking at," "looking upon" (xxiii). I agree that the customary translation can cause confusion, but I do not follow Carus's suggestion. Attempting to change this well-established practice now could cause more confusion.

> In whatever way and through whatever means a cognition may relate to objects, that through which it relates immediately to them, and at which all thought as a means is directed as an end, is intuition. (A19/B33)

Kant does not use the term '*anschauung*' to denote a certain type of knowledge, belief, or disposition to believe. It is true that sometimes, for instance, in the *Prolegomena* Kant refers to the judgements of mathematics as intuitive [*intuitiv]* (§7 4:281). But this is a derivative use. Intuition for Kant is the means through which we cognitively relate to objects. We can obtain knowledge *about* intuition, and that knowledge is immediately given and non-inferential. But intuition itself is an object of analysis in its own right. Kitcher offers an analogy to think about this.

> Let us imagine that we are condemned to look toward a surface, normally unlit, onto which pictures are periodically flashed. We can discern some order and pattern in the pictures by learning the geometrical properties of the surface […] We are able to draw luminous figures on the surface. We do so by following rules. (Kitcher 1975, 31)

Rules govern the formation of images on this surface. Our knowledge about intuition consists of knowledge of these rules, to which all mathematical knowledge reduces. This analogy is an excellent starting point. But it is also somewhat misleading, since it fails to highlight two important distinctions that Kant draws in the transcendental aesthetic.

Kant distinguishes *empirical intuition* and *pure intuition* (A20/B34). Through empirical intuition, sensations are given to us. A sensation can be given to us as a sensory perception, or when we imagine a sensory perception. Through pure intuition, pure representations are given to us. When we abstract away from all that belongs to sensation (Kant's examples are impenetrability, hardness, and colour in A21/B35), we are left with pure representations.

Kitcher's analogy might seem to suggest that we derive the rules of pure intuition by imagining shapes, but this is not quite what Kant has in mind. Hintikka (1984) argued that instead we should conceive of this "by means of what twentieth-century logicians would call instantiations" (101). We deduce the rules governing pure intuition not by imagining, say, the shape of a triangle—that would amount to a sensation—but rather by considering an arbitrary triangle (A141/B180).[15]

Kant distinguishes *sensibility* from *thought*: "objects are […] given to us by means of sensibility, and it alone affords us *intuitions*; but they are *thought* through the understanding" (A19/B33). Thought involves concepts and intuitions together, whereas sensibility involves no concept.[16] This suggests that we should really think in terms of three

[15] Here Hintikka has in mind the traditional understanding of 'arbitrary triangle': in a proof that involves an arbitrary triangle one considers a triangle and appeals only to properties that are known to belong to all triangles. Fine (1985), among others, proposed a more sophisticated view about arbitrary objects. But there is no need to discuss this in detail.

[16] In the *Jäsche Logic* Kant gives the following example:

> If a savage, for example, sees a house in the distance, the use of which he does not know, he has, no doubt, the same object presented to him as another man who knows it definitely as a habitation intended for men. But the form of this knowledge of the same object is different in the two men. In the one it is mere Intuition; in the other it is Intuition and Concept combined. (JL, §5)

This savage [sic] lacks the concept of house. Thus, looking at a house, the savage only forms a sensation, a sensory stimulus with no concept attached. On the other hand, we have a man [sic] that forms a thought upon seeing a house. This thought is composed by the sensation that the savage also has together with the concept.

translucent surfaces on top of each other. The outermost surface is understanding, which provides concepts. The intermediary surface is empirical intuition, which governs sensations, the outer layer of sensibility. And the innermost surface is pure intuition, which governs pure representations, the inner layer of sensibility.

This three-surfaces analogy should not be taken to imply that one of these surfaces is more important than the others: "thoughts without content are empty, intuitions without concepts are blind" (A51/B75). Objectively valid judgements must always involve both concepts and intuitions. As Hanna (2022) puts it: "intuitions and concepts are cognitively complementary and semantically interdependent *for the specific purpose of constituting objectively valid judgments*" (first supplement). Knowledge, for Kant, is always *about* something which is given to the knower through intuition.[17] But it qualifies as objectively valid only if the knower attaches the correct concepts to the sensation or pure representation.

This might seem somewhat puzzling. On the one hand, Kant says that all objectively valid judgements involve both understanding and intuition. On the other hand, he says that objectively valid judgements of mathematics are grounded in pure intuitions. This apparent tension is removed by noticing that pure intuitions are the source of justification for our arithmetical knowledge, but they are not themselves a sort of knowledge. Again, it would be

[17] One way to think of this employs Brentano's (1874) terminology. Knowledge for Kant is a mental state that invariably is *about* something or *represents* something; thus, it is what Brentano called an *intentional* mental state. Through empirical intuition minds relate to material objects or imagined material objects. Through pure intuition minds relate to objects that have none of the hallmarks of material objects, say, an arbitrary prime number.

a mistake to conceive of intuition [*anschauung*] as a type of knowledge, belief, or disposition to believe. In the transcendental aesthetic we acquire knowledge *about* intuition, and of course this knowledge involves concepts—specifically, it involves the concept of magnitude (B203). Thus, even somebody lacking mathematical concepts, for Kant, has access to the source of justification for objectively valid mathematical judgements. What this person lacks is *understanding*. This explains why Kant occasionally refers to the judgements of mathematic as intuitive [*intuitiv*]. The justification for any mathematical truth is, in a sense, always in front of our eyes. It takes effort to recognize a mathematical truth as such. But even somebody lacking mathematical concepts has access to the justification of all objectively valid mathematical judgements.[18] Now, having distinguished Kant's notions of understanding, empirical intuition, and pure intuition, I need to clarify (non-metaphorically) how each contributes to objectively valid mathematical judgements.

[18] I am talking generically about mathematical judgements here, although my focus is on arithmetical judgements. This should not be a problem because what I said so far applies equally to the judgements of arithmetic and geometry. The main difference between Kant's theory of geometry and his theory of arithmetic is this: our knowledge of geometry is grounded on the pure intuition of space alone (by which the mind relates to pure representations as phenomenologically external to it); our knowledge of arithmetic has a dual nature: "Pure mathematics deal with *space* in geometry, and *time* in pure mechanics. In addition [the concept of number, which is the concept treated in arithmetic] requires the assisting notions of time and space" (ID, 62 see also Prolegomena §10 4:283).

Let us begin with the faculty of understanding. I follow the exposition given at the beginning of the transcendental logic.[19] Kant says that just as there are empirical intuitions, there are also empirical concepts. He immediately sets aside these, on the ground that he is interested in *a priori* objectively valid judgements (A51/B75). Then, he says that the science of the general rules of the understanding is logic, and he goes on to distinguish various aspects of it. Logics of a *particular* use of the understanding study the rules of correctly thinking about a certain kind of object (A52/B76). These too he sets aside because he is not interested in any *particular* science. Logic is *applied* when it considers the psychological features of actual thinking agents (his examples include the laws of memory, the power of habit, and inclinations in A53/B77). This too he sets aside on the ground that the truths of mathematics have nothing to do with the subjective psychological features of individual knowers—that is, he is concerned with *ideal* knowers. This leaves us with *general* (non-particular), *pure* (non-applied) logic. This, as MacFarlane (2002) among others emphasises, is the Kantian notion corresponding to Frege's notion of logic.

Pure general logic, for Kant, is "a science of the necessary laws of thinking without which no use of the understanding and of reason takes place at all" (JL§1). This is a "negative condition for truth" (A59/B84) because it cannot *expand* our knowledge. As MacFarlane puts it: "[unlike Frege] Kant takes the generality of logic to preclude logic's having substantive content." (2002, 33).[20] This corresponds to Kant's narrow conception of logic:

[19] See also JL§2.

[20] MacFarlane argues—contra Wang (1957), among others—that the disagreement between Kant and Frege on the nature of logic is not verbal. It is in this context that he draws the distinction above.

the theory of syllogisms outlines rules on what can be derived from any two premises but sanctions no conclusion from null premises. By contrast, Frege's logic has theorems.

Objectively valid mathematical judgements meet the negative conditions of truth because no use of the understanding fails to meet them. But this cannot be all there is to them. There must be some other aspect of the faculty of understanding that contributes to mathematical knowledge because, among other reasons, the judgements of mathematic say something about the world. At this juncture, Kant introduces *transcendental logic.* This is the science studying concepts that ideal agents form based on pure representations. Kant emphasises that objects of pure thought must be given to us through intuition. Unlike pure general logic, transcendental logic is not empty (A62/B87). This is crucial for Kant's justification of mathematical knowledge. The concepts of space and time are derived from pure intuitions (B40-41 and B48-49)[21] because intuitions of space and time are, Kant says, "the conditions of possibility of objects as appearances" (B122).

Pure intuitions are the faculties through which ideal minds relate to pure representations. As ideal agents reason about pure representations, they realize that there are limits to this faculty. Kant's examples in "Axioms of Intuition" (B202-7) include "between two points only one straight line is possible." And, in a letter to Rehberg dated 1790, Kant gives an arithmetical example: for any arbitrary negative number *a, √a* can never be given to us (K*Corr,* 357).[22] These are all conditions of possibility for pure representations: if something is

[21] B40-41 is "The transcendental exposition of the concept of space" which is largely a summary of *Prolegomena §*6-9. B48-49 is "The transcendental exposition of the concept of time."

[22] Sutherland (2017), among others, takes for granted the "primacy of natural numbers" (148, footnote 1). Tait (2020), on the other hand, argues that Kant is primarily concerned with real

a pure representation, then it cannot be the square root of a negative number. In Kitcher's (1975) analogy, these are the rules governing the formation of shapes on the surface that we are forced to gaze upon. And, of course, these are also the principles of pure mathematics as Kant conceived it.[23] Thus, our knowledge of pure mathematics is nothing over and above our knowledge about what can and cannot be a pure representation. We learn this by running Euclidean-style proofs. In the doctrine of method, Kant gives as a concrete example the proof that the internal angles of a triangle always sum up to 180° (A716-7/B744-5). This starts by considering an arbitrary triangle and proceeds through a chain of inferences allowed by intuition to the desired conclusion; we would have to form an impossible pure representation to conceive a counterexample at any step of the proof.[24]

It seems clear which notion wears the trousers, according to Kant. General pure logic is a negative condition for truth. By contrast, the study of pure intuitions can lead to discoveries: we can discover new conditions of possibility for pure representations. In a sense, these

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numbers. For the time being I leave this debate aside. It hardly matters, for the present discussion, whether in the letter to Rehberg Kant is talking about numbers in a primary or secondary sense.

[23] Again, I am talking generically about pure mathematics. To be precise, Kant in "Axioms of Intuition" says that in geometry there are axioms, whereas in arithmetic "although various of [its] propositions are synthetic and immediately certain (*indemonstrabilia*), there are nevertheless no axioms in the proper sense […] for otherwise there would be infinitely many" (B204-6). This reflects the conception of axioms in Kant's time.

[24] Friedman (1985) maintains that drawings, or graphical imagination, are an essential part of geometrical proofs for Kant. I think about this in terms of instantiation, as Hintikka (1984) suggests, because if imagination played an important role in Kant's proofs, it would be difficult to explain why Kant sets aside sensation and empirical intuition. See also Hintikka (2020).

conditions are always in front of our eyes, but it is only when we form judgements about them that they come to constitute actual knowledge. We can also find new consequences for conditions of possibility that we were already aware of. But this is a derivative discovery. The source of the basic principles of mathematics are pure intuitions. And this finally leads us back to the secondary problem of pure reason.

So far, Kant taxonomized various faculties of ideal minds that contribute to their mathematical knowledge. But his goal is more ambitious than this. He also needs to show that ideal thinkers are *justified* in believing, among other things, that 7+5=12 (A84-B116-7). In principle, one can accept Kant's claim that the source of ideal agents' mathematical beliefs are the conditions of possibility for pure representation but deny that these beliefs are justified. To rule this out, Kant needs to tackle the secondary problem of pure reason: having reduced mathematical knowledge to knowledge about pure intuitions, he needs to show that knowledge of pure intuitions is justified.

The boundaries of conceivability

Let us begin with an interpretation that I ultimately do not endorse. Stroud (1968) suggests interpreting Kant's approach to the *quid juris* in light of the "refutation of idealism" (B274-9). This, he claims, amounts to thinking about analytic a priori judgements in terms of a *privileged class of propositions.* The truth of any proposition in the privileged class "follows from the fact that somebody asserted it, or denied it, or said anything at all" (Stroud 1968, 253). The stock example is 'there is a language.' Seemingly, the truth of this proposition is a necessary condition for somebody to say anything at all.

Stroud emphasises that it seems extremely difficult to show that any given proposition is a member of the privileged class. Even if we accept that the class has *some* members, Kant

still needs to show that all the truths of arithmetic are in it. And this faces a very hard challenge: "for any candidate S, proposed as a member of the privileged class, the skeptic can always very plausibly insist that it is enough to make language possible if we *believe* that S is true" (ibid. 255). Moreover, Stroud emphasises, members of the privileged class can be contingent truths—that there is a language is indeed a contingent truth. Thus, transcendental arguments fail to prove the necessity of mathematical truths, even if they are successful.

I agree that this gets close to what Kant says in the "refutation of idealism" (B274-9).[25] But we should not forget that this section was added to the second edition. Indeed, Kant

[25] I say that Stroud interpretation *gets close* to what Kant says because talk in terms of proposition and the limits of language is mostly alien to Kant. In "the refutation of idealism" we read:

> Theorem. The mere, but empirically determined, consciousness of my own existence proves the existence of objects in space outside me. Proof. I am conscious of my existence as determined in time. All time-determinations presuppose something *persistent* in perception. This persistent thing, however, cannot be something in me, since my own existence in time can first be determined only through this persistent thing. Thus the perception of this persistent thing is possible only through a *thing* outside me and not through the mere *representation* of a thing outside me. (B274-6)

Probably Stroud's analysis can be rephrased in terms that fit what Kant says here. In this interpretation, the proposition 'there are mind independent things' would be in the privileged class because it is entailed by the persistence of the subject in the Cartesian *cogito ergo sum.* And the difficulty—with which, as we shall see, Kant seems indeed to be struggling—would be to show that the subject's persistence truly entails that there are mind-independent things, rather than entailing that the subject *believes* that there are.

mentions it in the preface to the second edition as "the only thing I can really call a supplement" (footnote at Bxxxix). This strongly suggest that there was some earlier reasoning with which Kant meant to give a foundation to mathematics. My suggestion is that the earlier reasoning effectively presupposed the denial of Cartesian skepticism and that explains why Kant later added a section to refute Cartesian skepticism. Consider the following passage, which is left unchanged in the second edition:

> The supreme principle of all synthetic judgements is […]: Every object stands under the necessary conditions of the synthetic unity of the manifold of intuition in a possible experience. (A158/B197)

The necessary conditions of pure representations—which, as we saw, are the principles of pure mathematics—are the conditions under which *every object* can be given to us in a possible experience. Pure intuitions are somewhat irremovable. This is a point that Kant repeats time and again in both editions of the first critique. In the first edition of the transcendental aesthetic, for instance, we read: "In regard to appearances in general one cannot remove time, though one can very well take the appearances away from time" (A31). And in the second edition we read: "One can never represent that there is no space, though one can very well think that there are no objects to be encountered in it" (B38).[26]

[26] Examples can be multiplied: "All thought […] must ultimately be related to intuitions" (A19/B33), "time is the *a priori* formal condition of all appearances in general" (A34/B50). See also A107, A116, A127, B160-1, B165. Moreover, notice that in the letter to Rehberg mentioned above, Kant addresses the problem: how to account for knowledge about irrational numbers such as $\sqrt{2}$ without postulating that the understanding can represent objects independently of pure intuitions? (KCorr,

Ideal agents relate to the world looking through three translucid surfaces. Two surfaces can be removed (understanding and sensation), but the deepest one is irremovable. We can abstract away from it in pure general logic, but that amounts to thinking about nothing. Any thought that relates to the world must do so through pure intuitions. Thus, I suggest that according to Kant, thoughts whose objects fail to meet the conditions of possibility for pure representations are inconceivable:

Inconceivability

The thought T is *inconceivable* for a subject S if and only if S cannot assert T

I distinguish two senses of thinking a thought T: (i) *grasping* T and (ii) *asserting* T. In Frege's formal language, one asserts to a sentence by placing the judgement stroke before it, this amounts to sincerely asserting that the sentence is a name of the True (Frege 1893, §5). By contrast, one can grasp a sentence that one believes to be false, for instance by placing the horizontal stroke before it or by asserting to its negation. Given the quotations above, it might seem that Kant is thinking in terms of *grasping*. But, I suggest, he is really thinking about what is impossible to *assert for idealized human beings*. Before giving my evidence, however, I need to explain how these Fregean notions relate to Kant's terminology. Kant

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356). This further supports the idea that it is crucial for Kant that nothing can be given to us except what the mind relates to through pure intuitions. Irrational numbers might seem to be counterexamples to this: prima facie, we can think about $\sqrt{2}$ conceptually, but we cannot form a pure representation of it. Kant's way out of this is to claim that $\sqrt{2}$ is a partially spatial object. For instance, it is the size of the hypotenuses of a triangle whose catheti both have size 1. This might suggest that Kant's claim that arithmetic is grounded in both pure intuitions was a way to deal with this sort of case.

does not have an explicit notion corresponding to Frege's grasping. And, on the other hand, he recognizes several different notions that may correspond to Frege's asserting. I suggest that Kant's *thinking* is a counterpart of Frege's *grasping*, and that *scientific belief* is a counterpart of *asserting*. One may suggest that Kant's *understanding* or *cognizing* should be the counterpart of Frege's grasping. But Kant states that thinking for him is a broader notion than understanding or cognizing. One can think about, but not cognize, super-sensible things such as God or things in themselves (Bxxvi, footnote). Understanding is a faculty devoted to scientific knowledge, which for Kant, as for Frege, is marked by a distinctive way of grasping and asserting. But let us proceed in order.

Conant (1992) argues that, for Kant, what fails to meet the conditions of possibility of thought is impossible to grasp. This makes it also inconceivable in the sense above—one can never assert a sentence that one knows to be senseless, because there is no thought to assert. However, and this is crucial, this definition allows for meaningful thoughts to be inconceivable: one can never assert a thought that one fails to grasp, but one may grasp a thought that one never assert to.[27]

I agree with Conant that the principle of Kant's pure general logic—which is the principle of non-contradiction—is a condition of possibility for thought. Indeed, Kant writes: "I can think whatever I like, as long as I do not contradict myself, i.e., as long as my concept is a possible thought" (Footnote Bxxvi, See also JL 14-15). This suggests that, for Kant, sentences containing explicit contradictions are unthinkable in the sense that there is no thought to grasp in relation to them. One could have the impression that to a contradictory sentence

[27] I would like to thank [blinded for review] for suggesting me to draw this distinction.

corresponds a thought, say, because the contradiction is difficult to spot. In these cases, pure general logic comes to our aid as a *canon* for thought—this explains in what sense logic, for Kant, judges the correctness of our cognition.

On the other hand, it seems unlikely that Kant intended the conditions of possibility of pure representations to be a condition of possibility of thought. Most likely, Kant takes logical aliens to be impossible, but he acknowledges the possibility of intuitions aliens. Indeed, he writes:

> We cannot judge at all whether the intuitions of other thinking beings are bound to the same conditions that limit our intuition and that are universally valid for us (A27/B43). [28]

The Kantian notion par excellence of things going beyond the conditions of possibility for what can be given to us in intuition is, of course, the notion of *things in themselves*. And Kant clearly says that we can *think* objects as things in themselves (Bxxvi). Judgements

[28] As we saw, Kant conceives of ideal knowers as agents that are not limited by the psychological features of actual human beings, such as "the law of memory" (A53/B77 see also the note on top of page A27 in Kant's copy of the first edition). Since he accepts the possibility of intuitions aliens, we must further distinguish *human* ideal knowers from *alien* ideal knowers. Both are not limited by psychological features, but the formers are limited by pure intuitions and the latter are not. Note that the intuitions of alien ideal knowers should not be confused with God's intuitions. God for Kant has intuitions structurally different from ours because they are intellectual. Aliens, on the other hand, have intuitions of the same sort as ours, but they constraints their pure representations differently. It's a complicated problem to assess what Kant meant exactly by 'intellectual intuitions,' but I must leave it at that here.

about things in themselves not only have a sense, but according to Kant, we ought to believe in some of them. Kant distinguishes *moral belief* from *scientific belief*. Things in themselves can never be given to us, and this prevents us from having scientific beliefs about (cognize, understand) things in themselves (Bxxvi-Bxxxi). But we can, indeed we ought to hold some moral beliefs about things in themselves. As Hanna (2022) puts it: "moral belief philosophically trumps epistemic belief" (second supplement).[29]

Kant holds that there are several different ways in which we can take a thought to be true. For instance, we may believe a thought dogmatically, morally, or pragmatically. But the epistemic state that he is chiefly interested in in the first critique is *scientific* belief. In this respect, Kant's epistemology is analogous to Frege's. In *Über Sinn und Bedeutung* Frege emphasizes that we need to "abandon aesthetic delight for an attitude of scientific investigation" (Frege 1892, 163). When Frege talks about asserting he does not mean a somewhat thoughtless aesthetic asserting by which one could agree that "Odysseus was set ashore at Ithaca while sound asleep" (ibid. 162). Rather, he refers to an epistemic state in which one "seriously" takes a sentence to be true, or false (ibid.). When we are in this scientific mode, Frege continues, we must always ask whether the terms we use refer to something, and if yes, to what. Analogously, Kant in the first critique sets aside all epistemic

[29] Guyer and Wood, translate '*glauben'* as 'faith' at Bxxx: "I had to deny knowledge in order to make room for faith," although later, in a similar context, they translate it as 'belief' (A825/B853). Hanna maintains that translating '*glauben'* as 'faith' is misleading (2022, Second supplement). I set this debate aside. All that matters for the present discussion is that Kant takes judgements about things in themselves to have meaning, but he claims that we can never scientifically believe them, or their negation.

states except for scientific belief, and he argues that we can only have scientific beliefs about things that are given to us through intuition.[30]

The conditions of possibility of pure representations, according to Kant, set the boundaries of what we can scientifically believe. That is, in Fregean terms, we can only assert sentences that are about something given to us through intuition. This raises the question: how did Kant seek to prove that any thought is inconceivable for ideal human agents? As we saw, Kant thought that the boundaries of intuitions can be discovered by running Euclidean-style proofs. To reject any step of a valid proof one would have to assert an inconceivable thought. But this does not explain how a subject S can find out that what is inconceivable for S is inconceivable for *all* ideal human beings.

This question is extremely important, but addressing it here would lead me beyond Kantian exegesis and into the contemporary debate. Kant seems to have taken for granted that anybody would agree that there are no true contradictions and 7+5=12. We know that these claims can be denied by rational experts: Priest (1987) defends dialetheism and Field (1980) maintains that there are no mathematical truths. I remand to another time a discussion about how a Kantian solution to the secondary problem of pure reason could be made

[30] Dummett (1993) suggests that Frege's notion of *referring* is analogous to Husserl's *intentionality*—which is part of Brentano's legacy. I suggested that one way to think about Kant's notion of something *being given to minds* is in terms of intentionality. In this framework, Frege's remark that when we are in the scientific mode we must always ask about the reference of our terms assumes a Kantian overtone.

plausible nowadays.[31] From Kant's viewpoint it was not disputable that Euclid's proofs—among others—are valid. The problems, for him, were to show how this indisputability makes us justified in believing Euclid's conclusions, and how we can make discoveries with demonstrations. The latter problem, as we saw, he tackled by saying that although pure intuitions are always in front of our eyes, we may fail to form concepts about them. As for the former problem, we are finally at the position to see what Kant says.

Let us assume that ideal human agents cannot help but believe that 7+5=12—or, better, they cannot assert 7+5≠12. At first, Kant acknowledges that the step from inconceivability to justification is unwarranted, since he admits the possibility of things in themselves and intuitions aliens.

> Since we cannot make the special conditions of sensibility into conditions of the possibility of things, but only of their appearances, we can well say that space comprehends all things that may appear to us externally, but not all things in themselves, whether they be intuited or not, or by whatever subject they may be intuited (A27/B43).[32]

[31] Another problem is this: many contemporary logicians reject classical principles without asserting their negation, say by taking both to be truth value gaps. But Kant's solution to the secondary problem of pure reason, as we are about to see, presupposes that there are no truth value gaps. And indeed the same applies to Frege's and Wittgenstein's solution. Any attempt to modernize this approach would need to tackle this.

[32] See A34/B51 for the corresponding remarks about time.

Kant concedes that the inconceivability of a thought does not entail that its negation is true. This, however, is not presented as a setback on the road to the justification of mathematics. Rather, Kant goes on to draw his conclusion on the same page.

> The proposition "All things are next to one another in space" is valid only under the limitation that these things be taken as objects of our sensible intuition. If here I add the condition to the concept and say: "All things, as outer intuitions, are next to one another in space," then this rule is valid universally and without limitation. Our expositions accordingly teach the reality (i.e., objective validity) of space in regard to everything that can come before us externally as an object, but at the same time the ideality of space in regard to things when they are considered in themselves (A27/B43).[33]

Although he abstracts away from psychological limitations, Kant's solution to the secondary problem of pure reason is given from the viewpoint of human beings: as far as ideal humans are concerned, the principles of pure mathematics are true. We cannot exclude that things in themselves are not like things given to us, but we have good reasons to disregard this possibility. To seriously consider this would be, as Frege later puts it, "an attempt to jump out of one's own skin against which I can only urgently warn" (*Grundgesetze,* xvii). Indeed, considering this possibility amounts to considering a skeptical scenario in which ideal humans never possess knowledge.

Suppose that ideal humans can never assert 7+5≠12, but in fact in some circumstances 7+5≠12. Ideal humans could never identify the anomalous cases, since as far as they are

[33] See A35/B51 for the corresponding remarks about time.

concerned, the principles of pure mathematics are universally valid. This casts a doubt over all their beliefs involving collections of seven things together with collections of five things. And of course, this is an arbitrary example. Anything at all that is given to us in scientific belief is subject to the laws of mathematics. Therefore, in Kant's framework, doubting the laws of mathematics amounts to doubting the human epistemic standpoint as a whole.

This interpretation explains nicely why later Kant added a section to refute skepticism. If we assume that ideal humans can possess knowledge about the ontologically external world,[34] then Kant's reasoning makes them justified into believing the principles of mathematics. Kant initially took for granted that humans do possess some knowledge about the external world. But later he came to recognize this as a problematic assumption, and accordingly, he decided to add a supplement.

Kant was never quite satisfied with his refutation of skepticism. In the preface to the second edition of the first critique he recommends a first amendment (footnote Bxxxix), and then he recommends a second amendment in the "General Note on the System of Principles" (B288-94). Furthermore, after the publication of the second edition, Kant drafted several new versions of that argument, including R 5654 (18:305-13) and R 6311 (18:607). Given this, it is easy to see how one could always find oneself, as Stroud (2018) puts it, "at least

[34] Guyer (1983) discusses these and several other fragments concluding that Kant indeed intended his refutation to establish our knowledge about *ontologically*, rather than *phenomenologically* external objects. This use of 'ontological' is anachronistic. In A373 Kant distinguishes objects outside us in the *transcendental* sense (things in themselves) from external appearances or objects *empirically* external. I follow Guyer in using modern terminology here because, at this juncture, the contemporary notions seem to map nicely onto Kant's.

one step short of the illumination promised by [the] Kantian idealist doctrine" (181). Even if we accept that humans cannot help but believe that the principles of mathematics are universally valid, we seem unable to grant that they in fact are universally valid unless we can also prove that some of our beliefs constitute knowledge.

In the next section I argue that Frege, in his own framework, offered the same solution to the secondary problem of pure reason. The negation of laws of thought is inconceivable, and therefore, to doubt them would amount to doubting the human epistemic standpoint. Unlike Kant, however, Frege never attempted a refutation of skepticism. Rather, at this juncture he had the attitude of the earlier Kant: we should disregard skepticism, rather than attempting to refute it. It might be impossible to definitely prove that our epistemic standpoint is not structurally flawed, but as a matter of fact, our epistemic standpoint is all we have.

3. Frege's Solution

Frege's route to the secondary problem of pure reason is different from Kant's. Frege famously suggested that it may be possible to ground all truths of arithmetic on logic alone. Thus, for him, the problem calls for a justification of logic. According to Conant (1992), Frege held that sentences denying laws of thought are impossible to grasp. And Stroud (2018) replies that Frege really left the problem unsolved.

I suggest that Frege's solution was to claim that ideal humans cannot assert the denial of a law of thought, although they can grasp it. This is not what Kant said about logic, but rather what Kant said about pure intuitions. First, I outline the differences between Kant's theory of intuition and Frege's. Then, I argue that the notion of inconceivability I defined above is at play in Frege's solution to the secondary problem of pure reason.

The Logocentric Predicament and the Togetherness Principle

Ricketts (1985) quotes Sheffer's (1926) review of *Principia Mathematica:*

> The attempt to formulate the foundation of logic is rendered arduous by a […] "logocentric" predicament. *In order to give an account of logic, we must presuppose and employ logic* (Sheffer 1926, 228).

And he continues:

> Inquiry into any subject matter whatsoever draws implicitly on [the principles of logic] and must conform to these standards on pain of incoherence. Logic then has a privileged position among the sciences (Ricketts 1985, 3).

As we have seen, Kant held that we are in the logocentric predicament. The principle of non-contradiction, for Kant, is a condition of possibility of thought. That is, there can be no contradictory thought—or better, there can be no contradictory employment of understanding. This makes logic a central but somewhat empty discipline; a body of rules rather than a body of truths. The negation of a logical principle is not a thought. Thus, logical principles too are somewhat contentless; they tell us how to formulate proper thoughts but they themselves say nothing.[35]

MacFarlane (2002) argues that Kant and Frege agree that we are in the logocentric predicament—in his terminology, they agree that logic is *maximally general*—but they

[35] This is problematic. Kant seems to be saying at once that the principle of non-contradiction says nothing, and that it says something about all thoughts. This can be solved by noticing that Kant, if only implicitly, distinguishes procedural and declarative knowledge. A logician, for Kant, knows how to discern bogus sentences from thoughts but she possess no propositional knowledge about logic.

disagree on what this entails. I am inclined to accept that this is not a verbal disagreement. But this hardly matters for the present discussion. What matters is only that according to Frege logic is maximally general, but not contentless. While Kant says that we can abstract away from pure intuitions only in the sense that we can think about nothing, Frege claims that there are some substantive subject matters that we can think about once we abstract away from pure intuition.

> Empirical propositions hold good of what is physically or psychologically actual, the truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest vision of delirium [...] remain, so long as they remain intuitable, still subject to the axioms of geometry. Conceptual thought alone can after a fashion shake off this yoke, when it assumes, say, a space of four dimensions or positive curvature. (Frege 1884, §14)

According to Kant "thoughts without content are empty, intuitions without concepts are blind" (A51/B75). Frege, by contrast, thinks that conceptual thought can stand on its own. Some object can be given to us—or, in Frege's terminology, we can *refer* to some objects[36]—even when we disregard pure intuitions. That is, Frege rejects what Hanna (2022) calls *the togetherness principle*—this is Kant's idea that sensibility and understanding are

[36] As Dummett (1993) argued, Frege seems to have given to a feature of linguistic items (reference) the role that Brentano gave to a feature of mental phenomena (intentionality). This provided Frege with a clear methodology to study how objects are given to us: since objects are given to us through reference, a study of language can reveal the necessary conditions for objects to be given to us. See also Smith (1989), who despite disagreeing with Dummett's interpretation of Husserl, accepts that reference plays this role for Frege (9-10).

distinct but cognitively complementary mental functions. Indeed, Frege writes "I must […] protest against the generality of Kant's dictum: without sensibility, no object would be given to us (1884*,* §89).[37]

[37] Further evidence that Frege had a non-Kantian conception of intuition can be found in what he says about the analytic-synthetic distinction. In the first footnote of the *Grundlagen*, Frege says that he does not intend to assign a new meaning to these terms (1884,§3). But later he says that Kant defined 'analytic' "too narrowly" and that he, Frege, uses the term in a "wider sense" (ibid.§88). As Kenny observes, the later assessment seems to better fit also what Frege says earlier (1995,56-7). Indeed, Frege writes: "[if] it is impossible to give [a] proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the [proved] proposition is a synthetic one" (1884,§3). This echoes what Kant says about logics of a particular aspect of the understanding (A52/B76). When Frege says that he agrees with Kant that geometrical truths are synthetic (1884,§89), he really means that geometrical truths can be derived from the principles of some special science. Thus, the agreement with Kant at this juncture is illusory. According to Frege, intuitive principles are not synthetic in a Kantian sense. For him, the hallmark of synthetic statements is that they can be derived from the principles of special sciences, because having rejected the togetherness principle, nothing is left other than general logic and special sciences to derive them from. Later in his life, having abandoned logicism and endorsed a Kantian view about arithmetic, Frege will also come to agree with Kant on the nature of intuition and synthetic judgements (Frege, 1924), but in this phase he has a non-Kantian understanding of intuitions.

Frege thought that intuitions are a *particular* way in which objects can be given to us.[38] In logic, objects can also be given to us, if we do not refer to any *particular* object. The truths of arithmetic, Frege hopes to show,[39] can be proven by reasoning about objects in this abstract way; by reasoning about arbitrary objects without taking into account the special features of any particular object.[40]

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[38] Frege seems to have thought that intuitions in general are analogous to what Kant calls *empirical* intuitions. Indeed, he argues against Kant's view by pointing out that we cannot picture 37863 fingers to ourselves all together (1884, §6) and that even those who think that the smaller numbers (nought and one) are intuitable must admit that they cannot be given 1000^{1000} in intuition (ibid.§89). This suggests that he thought of intuition as analogous to visual imagination, rather than the general function through which all objects—including those we cannot picture—are given to us.

[39] I say that Frege *hopes* to show this because he says that the arguments he gave against rationalism and empiricism in the *Grundlagen* are meant as preliminary evidence, only the completion of his lifelong project, he says, could give a definitive proof that arithmetic is analytic (1884, §90). Strictly, this is incorrect. He only needed to derive the Peano axioms, or other axioms from which the rest of arithmetic is derivable. But of course, at the time this had not yet emerged.

[40] MacFarlane (2002) suggests that, for Frege, logic is general because its laws imply norms for thinking about anything at all. This is compatible with the idea that laws of logic are about the special features of something in particular (Frege's examples are "negation, identity, subsumption, subordination of concepts" in Frege 1906b, 338). This is not compatible, though, with the idea that Frege's laws of thought are about the special features of particular *objects*. As MacFarlane puts it: "the laws of arithmetic yield norms for our thinking insofar as it is about what is numerable. But this turns out to be no restriction at all, since *(on Frege's view) the numerable is just the thinkable*" (2002, 38 emphasis mine). Identity, according to Frege, is a particular relation studied by logic, *and* it

> The basis of arithmetic lies deeper, it seems, than any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought? (Frege 1884, §14)

Frege, like Kant, aims to give a justification to arithmetic. This, for Frege, consists in finding the proof for every arithmetical truth and "following it up right back to the primitive truths" (Frege 1884, §3).[41] These primitive truths, he wants to show, are the "laws of thought"—that is the principles of logic conceived as general but not contentless. This leads us to Frege's version of the secondary problem of pure reason. If arithmetic stands in need of a justification, and this justification is to be sought in logic, then one might ask, does logic also stand in need for a justification?

Frege seems compelled to answer this positively. Indeed, he says "it is the nature of mathematics always to prefer proof, where proof is possible, to any confirmation by induction" (Frege 1884, §2). Given this, he should presumably try to prove the laws of

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is distinctively logical because the laws of identity apply to anything regardless of the special features of particular objects, unlike the laws of physics that only apply in a certain domain (see also Frege 1892).

[41] This idea recurs throughout Frege's writing. In the *Begriffsschrift*, Frege says that he aims to see "how far one could get in arithmetic […] supported only by the laws of thought, which transcend all particulars" and he adds—making it quite clear that he already has in mind his logicist project—that he developed his conceptual notation to make sure that, in the deduction of arithmetic, "something intuitive could not squeeze in unnoticed" (Frege 1879, 104). See also Frege 1897.

thought. But this is rendered arduous by the logocentric predicament. Any purported justification of logic would need to presuppose logic. Considering this, we could expect Frege to say that the principles of logic are only inductively confirmed, because a proof is impossible. But this does not seem to be what Frege has in mind, or else arithmetic in turn would only be inductively confirmed. Frege's commitments seem to pull him in two different directions here. To solve this dilemma, commentators tend to quote a passage in the foreword to the *Grundgesetze*—or parts of it.

The Controversial Passage

Let us first consider the passage in full, I add line numbering for ease of commenting.

1 As to the question, why and with what right we acknowledge a logical law to be true, logic
2 can respond only by reducing it to other logical laws. Where this is not possible, it can give
3 no answer. Stepping outside logic, one can say: our nature and external circumstances force
4 us to judge, and when we judge we cannot discard this law—of identity, for example—but
5 have to acknowledge it if we do not want to lead our thinking into confusion and in the end
6 abandon judgement altogether. I neither want to dispute nor to endorse this opinion, but
7 merely note that what we have here is not a logical conclusion. What is offered here is not a
8 ground of being true but of our taking to be true. And further: this impossibility, to which
9 we are subject, of rejecting the law does not prevent us from supposing beings who do so
10 [logical aliens]; but it does prevent us from supposing that such beings do so rightly; and it
11 prevents us, moreover, from doubting whether it is we or they who are right. At least this is
12 true of myself. If others dare in the same breath to both acknowledge a law and doubt it,
13 then that seems to me to be an attempt to jump out of one's own skin against which I can
14 only urgently warn. Whoever has once acknowledged a law of being true has thereby also

15 acknowledged a law that prescribes what ought to be judged, wherever, whenever and by
16 whomsoever the judgement may be made. (Frege 1893, xvii)

This is unusually evasive for Frege. In the first two lines, he seems to be saying that a justification of logic cannot be given. We are in the logocentric predicament, thus if a justification of logic cannot be given in logic, it cannot be given at all. On line three, however, Frege suggests that one *can* step outside logic and, in doing so, one could try to justify logic by observing that we cannot discard laws of logic. On line six, immediately after giving us this argument, Frege takes it back: the argument does not give us reason to think that the laws are true, it merely explains why we take them to be true. Moreover, this argument cannot exclude the possibility of logical aliens. On line ten, Frege gives us back the argument that he just took away. "This impossibility [human] of rejecting the law" cannot prove that the laws of thought are true or that there can be no logical aliens, "but it prevents us from supposing that [logical aliens are right]." Frege endorses this—creating a tension with what he said on line six—and he goes on with a vivid metaphor: doubting a law of logic is an attempt to jump out of one's skin. The last sentence is meant to illustrate this metaphor but as it turns out it is phrased quite ambiguously. Frege's use of the verb 'acknowledge' there can be read in two ways: 'if one *knows* that a law is true, then one *knows* that everyone ought to believe it' or as 'if one *believes* that a law is true, then one *believes* that everyone ought to believe it.'

First, I need to stress what I do not take to be controversial. Frege regards laws of thought as *weakly normative*. As MacFarlane (2002) puts it: "On Frege's view, [laws of logic and laws of physics alike] are descriptive in content but have prescriptive consequences" (37). In the pages preceding the controversial passage, Frege illustrates this very clearly:

> Every law stating what is the case [descriptive] can be conceived as prescriptive, one should think in accordance with it, and in that sense it is accordingly a law of thought. This holds for geometrical and physical laws no less than for the logical (Frege 1893, xvii).[42]

Conant (1992) and Stroud (2018) agree that Frege's laws of thought are not *about* how humans should think. The controversial passage is not about the *content* of laws, but rather their *justification*. Having argued that logical laws are descriptive claims about the world, not prescriptions about minds, Frege asks how we are justified in believing them. Here Conant says that Frege is self-defeating, and his opponents claim that Frege takes back the question of justification he just posed.

I agree that Frege is taking something back in the controversial passage. He is a logician, and he says that the question cannot be answered by logic (line two). Then he sketches a non-logical answer, but he adds that he neither disputes nor endorses this answer (line six). But it would be too quick to conclude, as De Pierris (1988) puts it, that "Frege leaves our knowledge of logic and its objectivity without justification […] because logic itself cannot give the answer" (310).[43]

Throughout his life, Frege is deeply concerned with the truth of the basic laws. In his reply to Russell's famous letter, Frege writes "your discovery of the contradiction […] left me thunderstruck, because it has rocked the ground on which I meant to build arithmetic. It

[42] See also Frege 1918, 351.

[43] Notably, De Pierris only quotes the first half of the controversial passage, just like Hacker does, among others (Hacker 1986, 43).

seems that […] my law V is false" (FCorr, 131). Famously, even before Russell's discovery of this contradiction, Frege had already expressed doubts about basic law V (1883, vii). And long after that, Frege he still wrote: "we cannot accept a thought as an axiom if we are in doubt about its truth" (1914, 205).[44]

Stroud (2018) emphasizes that one can be committed to the truth of a law and maintain that no justification can be given to it. But this attitude—which Stroud presents as his own—can hardly be attributed to Frege. This would amount to saying that Frege, for a time, thought that basic law V was true, and that the question as to how a subject can be justified in believing basic law V is unanswerable: one cannot say what would make a subject justified in believing it. But this suggests that Frege recognized no objective criteria for accepting or rejecting basic law V, which is plainly false. Upon receiving Russell's letter, Frege seems to have formed the belief that basic law V is false. Therefore, he recognized at least one necessary condition for accepting it—namely, if a thought is a basic law, then it entails no contradiction.

One could reply that this observation only shows that basic law V was not basic after all. Perhaps what Frege truly took as basic was the principle that no contradiction is true, which entails—given weak normativity as a bridge principle—that one must reject thoughts that one knows to be contradictory. I am sceptical about this, because I am inclined to take Frege at face value when he says that basic laws are basic. But let us consider this interpretation anyway: Frege gives a justification of the basic laws—to deny them would be

[44] By 1906 Frege gave up hope of rescuing basic law V from Russell's paradox (see Frege 1906a). Thus, Frege seems concerned with the truth of the laws of logic independently of the prospects of his logicist project.

contradictory—but he appeals to self-evidence to justify non-contradiction. Now, there is a grain of truth in the idea that Frege appeals to the principle of non-contradiction to justify some basic laws. For instance, he introduces basic law I as follows.

> According to §12
>
> 
>
> Would be the False only if Γ and Δ were the True while Γ was not the True. This is impossible (Frege 1893, §18).

Basic law I could be false only if Γ were true and not true. This is impossible; therefore, basic law I must be true. This is a reductio ad absurdum. But it is hardly an appeal to unexplainable self-evidence. Frege refers to §12, which is the section where he introduced the conditional stroke. There, Frege says that the value of a conditional "shall be the False if the True is taken as the [antecedent], while any object that is not the True is taken as [consequent]" (1893,§12). Now, this employs concepts that Frege introduced in the sections above. The horizontal and judgement strokes, negation, and the idea that no function can refer to both to the True and to the False are presented in sections §1 to §6, where Frege works out these notions informally with references to *Über Sinn und Bedeutung.* For instance, Frege writes "the value of the function $\mathfrak{E}^2 = 4$ is either the truth-value of the True, or that of the False [footnote: I argued for this in more detail in my essay *Über Sinn und Bedeutung]*" (§2). The impossibility to which Frege appeals in §18—that Γ cannot refer to both the True and the False—is an upshot of his theory of reference. Basic law I must be true because of how Frege defined the notions at play in it, and these notions had to be defined this way to fit Frege's theory of reference.

One could ask, then, what justifies the theory of reference itself? If in *Über Sinn und Bedeutung* Frege appealed to self-evidence, that could vindicate the self-evidence interpretation. But, in fact, Frege never appeals to self-evidence there. What he assumes are claims that he takes to reflect fully uncontroversial facts about scientific language. For instance, he assumes that in a scientific language, we demand that every term has a reference.[45] Frege seems to have taken for granted these claims– just as Kant took for granted what he took to be basic facts about human minds such as 'no human in their right mind could deny that 7+5=12.' Never mind whether these claims are true, or justified, that would be the topic for an entirely different discussion. What matters here is that Frege—and Kant—did not need to postulate a notion such as self-evidence to support their basic commitments. They thought that they could support them a priori with their own familiarity with the human mind and scientific language.[46]

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[45] I say *scientific* language because, as we saw, Frege sets aside cases in which we use language for "aesthetic delight" in favour of "an attitude of scientific investigation" in which we demand that no term fails to refer (Frege 1982, 33). On this ground, Frege postulates that no terms in his *Begriffsschrift* fail to refer, since it is a language designed for science (Frege 1983*,* §5 third footnote).

[46] It might seem psychologistic of Frege to appeal to his knowledge of his own mind. But Frege—and Kant—thought that only the study of individual minds belongs to psychology. Frege writes:

> Nothing would be a greater misunderstanding of mathematics than making it subordinate to psychology. Neither logic nor mathematics has the task of investigating minds and contents of consciousness owned by individual men. Their task could perhaps be represented rather as the investigation of *the* mind; of *the* mind, not of minds. (Frege 1918, 368-9)

This strongly suggests that Frege is reasoning about the thought processes and language of ideal humans, to which he takes himself to have a direct access—albeit, of course, a fallible access.

In general, Frege never talks in terms of self-evidence in the sections where he introduces the basic laws (1893, *Exposition of the concept-script*, §1-52*)*. On one occasion, Frege uses '*selbstverstandlich'* referring to the sentence *a=a* (§49). But here too Frege refers to his informal discussion about the equality sign, and he goes on giving a proof that *a=a*. Thus, this hardly supports the self-evidence interpretation. If this interpretation were correct, we would expect Frege to introduce the basic laws by claiming that they are unexplainably true. But, instead, case by case Frege gives explanations by working out informal arguments to the conclusion that the laws are true.

If we look at other parts of Frege's corpus, on the other hand, it is possible to find reasons to endorse the self-evidence interpretation. Famously, Frege uses the word '*einleuchten*' in connection to basic laws V after learning about Russell's paradox (Frege 1903, appendix 235-65).[47] And De Pierris quotes a 1914 article:

> Science demands that we prove whatever is susceptible of proof and that we do not rest until we come up against something unprovable. It must endeavour to make the circle of unprovable *primitive truths* as small as possible, for the whole of mathematics is contained in these primitive truths as kernel. (Frege 1914*,* 204-5).

This passage can be read as saying that logical axioms are self-justifying. The problem is that this hardly fits the methodology that Frege actually uses to discuss his basic laws. And it hardly fits what Frege says in the controversial passage itself; he does not endorse the

[47] See also Frege 1906a where Frege writes that Schoenflies's version of basic law V is not "as self-evident as one would wish for a law of logic" (182).

reasoning given "stepping outside logic" but that is reasoning nonetheless, not an appeal to self-evidence.[48]

I suggest explaining away this tension by distinguishing two phases of Frege's thought concerning the secondary problem of pure reason. After he learned about Russell's paradox, Frege started mentioning self-evidence more often. And then, gradually, talk in terms of self-evidence became quite common in his writings. This explains why interpreters defending the self-evidence reading tend to quote later works. On the other hand, if we look at what Frege wrote before learning about the Russell paradox, we find few statements that can be interpreted as appeals to self-evidence.[49] In the *Grundlagen*, for instance, Frege says that the laws of logic "neither need nor admit proof" (1884, §3). This might be an appeal to self-evidence, but Frege might also be making the weaker claim that one cannot *formally prove* logical axioms, although one can "step outside logic" to give rational reasons to believe them. The latter reading seems to better fit what Frege writes—using German rather than his formal language—to illustrate why his basic laws cannot be false.[50]

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[48] Notice also that the correspondence with Russell concerning basic law V is hardly a discussion on whether basic law V is self-evident .

[49] Incidentally, Russell seems to have had a similar, but more explicit, change of opinion. In *Principia Mathematica* he and Whitehead maintain that self-evidence is not indispensable to knowledge (Russell and Whitehead 1910, 59). But later Russell explicitly commits to the self-evidence solution (Russell 1913, 492)

[50] Another example is this

> In arithmetic […] we must demand that any assertion that is not completely self-evident [*selbstverstandlich*] must actually be proven (1903, §60).

There are reasons to doubt that even the later Frege resolutely endorsed the self-evidence view. This view seems to create problems with other Fregean ideas. It is natural to conceive self-evidence as a psychological notion. Surely this is how Russell (1913, 492) and Wittgenstein (WSM, 270) conceived it. On this reading, the self-evidence view makes Frege appeal to a psychological notion to justify basic laws. This does not square with Frege's rejection of psychologism, to which he was committed throughout his life. Of course, one might understand self-evidence in non-psychological terms. Dummett (1980) suggests that Frege held *epistemological platonism*—the view that we have an awareness, analogous to

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Out of context, this might seem a claim about the justification of basic laws. But, in fact, here Frege is criticising the habit of some mathematicians of omitting important steps in their proof and giving circular or vague definitions. Frege is not talking about basic laws here. Accordingly, it seems difficult to read this as a passage about the secondary problem of pure reason. More likely, Frege is making the point that one must omit no step of one's proofs, unless it is absolutely clear what the missing step would be—as he sometimes does, for instance, omitting explicit steps to remove redundant parentheses. The word '*selbstverstandlich'* appears as well in the previous section. Here even out of context the self-evidence interpretation seems off. Frege writes:

> It should strike one as entirely self-evident that a word should not be explained by means of itself, for otherwise it is treated in one breath as both known and unknown *(1903*, §60).

What is said to be self-evident here is the claim that one should give no circular definitions, which is hardly a basic law. And even in this case self-evidence is not a standalone epistemic concept, as Frege's later writing might seem to suggest, because Frege gives a reason why one should give no circular definition—the reason being that circular definitions contradictorily entail that the definiens is both known and unknown.

sense perception, of mind-independent objects in realm of thought.[51] But this only raises more questions. Russell (1912) held epistemological platonism in *The Problems of Philosophy,* and yet he conceded that self-evidence is a psychological notion. Therefore, this view is not automatically anti-psychologist.[52] Moreover, Dummett's interpretation is itself controversial. Indeed, De Pierris argues that Frege never worked out an explicit epistemological account of self-evidence. It seems, then, that if Frege really held the self-evidence view, he did not clearly spell-out how to reconcile it with his anti-psychologism.

[51] This is also fails to fit what Frege wrote in the *Grundgesetze.* Nowhere there does Frege say that he can perceive the basic laws in the third realm. Therefore, Dummett's is also an interpretation of the later Frege.

[52] *Ontological platonism*—the view that logical laws are true in virtue of properties of mind-independent entities—grants that the truth of the laws does not ontologically depend on mental facts. In "Thought" Frege explicitly commits himself to this (1918, 363). But ontological platonism is compatible with a psychologist epistemology. Crucially, it seems natural to admit, for epistemological platonists, that our awareness of the third realm is biased by our mental makeup. And this in turn makes the commitment to ontological platonism seem a *petitio principi*: logical laws are ontologically mind independent, but whatever law one might accept is justified on psychological grounds. This is the sort of tension that Wittgenstein alludes to when he criticizes Russell's and Frege's appeals to self-evidence—*alleged* in Frege's case. Now, I am not suggesting that this issue cannot be solved by a properly worked out platonism, but rather that it is unclear how Frege meant to solve this, *if* he held epistemic platonism.

In general, passages where the later Frege seems to endorse the self-evidence view are often brief and elusive.[53] Thus, it is unclear whether even the later Frege held this view. And surely, he did not hold it in the *Grundgesetze*, which is the context of the controversial passage. The question is, then, what does Frege take himself to be doing when he gives informal arguments to justify the basic laws?

This leads us to Conant's interpretation, according to which Frege gave a justification for the laws of logic. Specifically, Conant argues that Frege held a view that he—correctly—attributes to Kant. Laws of logic are a condition of possibility for thought—if something violates them, then it is not a thought. This is problematic, I suggest, because the secondary problem of pure reason presents itself in different guises for Kant and Frege. If we are to

[53] Consider Geach's translation of the famous sentence in the appendix to the Grundgesetze: "I have never disguised myself from [Axiom V's] lack of self-evidence" (1960, 234). As Ebert and Rossberg observe, the word '*einleuchten*' is not a technical epistemological term in German—they translate it as 'obvious' instead. And even in Geach's translation, this is hardly conclusive evidence for the self-evidence interpretation. Rather, the passage suggests that, for Frege, the feeling that a thought is self-evident is an indicator that it could be a law, but there are other more reliable markers—hence for a time he thought that axiom V was a basic law although he always felt that it is not self-evident. De Pierris's quote is similarly ambiguous. There, Frege might be making the claim that the primitive truths of arithmetic cannot be *formally* proven, this is compatible with the idea that they can be given an informal justification. One could expect to find Frege's later view in a 1924/5 manuscript "Sources of Knowledge of Mathematics and the mathematical natural Sciences." But this piece makes no mention of self-evidence. Indeed, it seems hard to distil *any* view on the justification of the basic laws from this manuscript.

find an historical antecedent to Frege's solution, we should not look at what Kant says about logic, but rather at what he says about pure intuitions.

As we saw, Kant thought that "I can think whatever I like, as long as I do not contradict myself, i.e., as long as my concept is a possible thought" (Footnote Bxxvi). That is, the principle of non-contradiction, for Kant, is a condition of possibility for thought; one can *say* that a contradiction is true, but one cannot *think* a contradiction. In Frege's terminology, this is to say that no thought corresponds to contradictions; thus, they are neither true nor false. One might assert a contradiction only by mistakenly associating a thought with a sentence that really has no meaning.

If Frege also held this view, then logicism as he conceived it was doomed to failure regardless of basic law V. Assume that contradictions are defective sentences to which no thought corresponds. Then, since the negation of gibberish is also gibberish, their negation is also defective. But the negation of a contradiction is a logical truth. Thus, with relatively basic assumptions, the view that logical laws define the boundaries of thought entails that logical laws lack propositional content.[54] This would force Frege into a contradiction, unless he rejected semantic compositionality, his theory of reference, or the idea that the basic laws are true. But these are all basic ingredients of Frege's logicism!

This should not be mistaken for an argument against Conant's interpretation. Conant is perfectly aware that his reading creates a tension between Frege's views. He holds, though,

[54] Indeed, Kant seems compelled to hold that we can only have procedural knowledge or understanding of logical laws, because they lack what we may now call propositional content. See footnote 35.

that this is the most plausible interpretation given what Frege says. To argue against this, I need to propose an alternative interpretation and give my reasons to think that it is overall more plausible.

On lines eight and nine of the controversial passage, Frege mentions an "impossibility" to which humans are subject. And, on the previous page, he says that, should he encounter a logical alien, he would say: "here we have a hitherto unknown kind of madness" (*1893,* xvii). This echoes a passage from the *Grundlagen:* "we have only to try denying any [law of arithmetic], and complete confusion ensues. Even to think at all seems no longer possible" (1884, §14). Conant's reading is that these passages say that the boundaries of logic coincide with the boundaries of thought. If we try to grasp the denial of a law of thought, confusion ensues because we are attempting to grasp something that is not a thought at all. That would be madness.

MacFarlane (2002) suggests an alternative interpretation for Frege's claim that complete confusion ensues as soon as one tries to deny a truth of arithmetic. This can be adapted to explain why Frege thought that no human in their right mind can take a contradiction to be true. Suppose that a subject believes a thought P, and she also believes ~P. Now, by disjunction introduction, the subject can form the belief P V Q, where Q is any thought whatsoever. Then by disjunction elimination the subject can prove Q. Since Q can be any thought, this shows that—under Frege's assumptions—accepting a contradiction renders our standards for reasoning incoherent. This is the logical principle *ex falso quodlibet*.

In this reading, confusion ensues not merely by *grasping* a contradiction, but by *asserting* it. Humans are bound to accept logical laws.[55] For instance, we are bound to accept basic law I because it could only be false if a thought were true and false at the same time, and this is not possible given Frege's background theory of reference. Now, given this, should we deny any logical law we would find ourselves believing a contradiction—that is, believing both the law, because we must, and its negation. But the logical laws that we are bound to accept entail *ex falso quodlibet*. Therefore, we have only to deny any law of logic to find ourselves compelled to accept that any thought is true. Of course, one might fail to realize this. Psychological logicians, according to Frege, are exactly in this predicament. But as a matter of fact, denying any logical law implies triviality, absurdity, madness.

In short, I suggest that the impossibility that Frege mentions in the controversial passage is an impossibility of *asserting* the denial of logical laws, rather than *grasping* it. Contradictions have a sense, but humans must reject them on pain of triviality. I have two reasons to prefer this interpretation.

First, this reading fits better what Frege says in the crucial passages. On line nine, Frege clarifies that the impossibility, to which we are subject, is an impossibility "of *rejecting*" [*zu verwerfen*] logical laws. And, analogously, in the *Grundlagen* he says that we have only to try "*denying*" [*leugnen*] a law to enter a state of confusion. Now, the most natural reading for

[55] Notice that here Frege is presupposing that if P cannot be asserted, then its negation must be. Frege (1919) presupposes a now controversial view about negation (Rumfit 2000; Ripley 2011). See footnotes 31 and 73.

'rejecting' and 'denying' a law seem to be asserting to its negation, rather than merely grasping it.[56]

Frege seems to follow this natural pattern of use. In the *Grundgesetze,* he uses '*verwerfen'* sparingly—except for the controversial passage, he only rarely uses it in volume two. But when he does use it, the word is always connected to views that he takes to be false, namely Heine's and Thomae's views and generally, the view that "piecemeal definitions" are acceptable in mathematics (§58, §60, §65, §100, and §121).[57]

I have no principled reason to exclude that Frege might have committed to views that entail a contradiction. But in the case at hand this would amount to saying that where he wrote "rejecting a law" he really meant 'grasping the negation of a law' and this seems highly unlikely. Throughout his life, Frege emphasises the distinction between grasping a thought and asserting to it. Indeed, each of his lengthier technical works begins by setting the distinction between the judgment stroke and the horizontal, or content stroke.[58] It seems

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[56] I would like to thank [blinded for review] for making this point during our discussion concerning the *Grundlagen* passage, and for pointing out that the claim that contradictions are meaningless, under Frege's assumptions, entails that logical truths are meaningless too.

[57] Analogous considerations apply to how he uses '*leugnen'* in the *Grundlagen*. For instance, Frege writes:

> It would also be wrong to deny [*leugnen*] that existence and oneness can ever themselves be component characteristic of a concept. What is true is only that they are not components of those particular concepts to which language might tempt us to ascribe them. (1884*,* §53)

Here the word is quite clearly used in connection to truth, rather than meaningfulness.

[58] See also Frege 1906c, 184 and 1906d*,* 185.

most unlikely that Frege could have been led to a contradiction by confusing the very notions that he kept distinguishing. And this is even more evident when we consider that in the controversial passage the word '*verwerfen*' occurs three times.[59] This suggests that Frege resolutely intended to use the verb 'to reject' there; thus, we should take him literarily.

Second, this interpretation fits better the methodology with which Frege introduces his basic laws and justifies them informally. Consider again how Frege introduces basic law I in 1893 §18. Frege invites the reader to grasp the law, by writing it with a horizontal stroke at its left. Then he observes that the law could only be false if Γ were true and false. This is impossible, and accordingly, Frege asserts the law by writing it with the judgement stroke at its left. The impossibility at play here is of the law being false. In no way is Frege suggesting here that he takes the negation of the law to be meaningless, rather he is saying that the law is impossible to reject, given how he defined the notions at play—given his theory of reference.

In the *Grundgesetze* Frege never introduces a formal symbol, or informal technical terminology denoting meaningless strings of symbols. On the contrary, he insists that—

[59] This repetition is hidden in the translation. Ebert and Rossberg translate the word as 'discard' on line four of the controversial passage. And, on line nine, where Frege uses it again, they edited it out writing "beings who do so" rather than "beings who reject it" [*die es verwerfen*]. This does not create a loss in meaning, since in the context 'reject' and 'discard' convey analogous meanings. But it should be noted that in the controversial passage Frege uses three times a word that he otherwise uses sparingly. This creates an emphasis that is lost in the translation, although literal meaning is preserved.

unlike natural language—his *Begriffsschrift* contains no names that fail to refer. Crucially, having introduced the horizontal stroke, Frege writes: "Names without reference must not occur in concept-script" (1893, §5 third footnote). This entails, given Frege's theory of reference, that all names in the *Begriffsschrift* have a sense—because a name's sense is the way in which it displays its reference. And this, in turn, entails that all sentences in the *Begriffsschrift* have a sense—because sentences are names of the True or the False. If Frege truly intended to justify the basic laws by saying that their negation is meaningless, he did so rather cryptically, since his technical vocabulary does not allow him to say such a thing.

Admittedly, one could reply that this is what Frege means when he says that we would need to "step outside logic" to justify logic. Perhaps Frege thought that although he cannot *say* that the negation of basic law I is meaningless, he can *show* it. But, again, there is an alternative interpretation. In the introduction to the *Begriffsschrift*, Frege famously compares his formal language with a microscope and natural languages with the human eye (1879, 105). The former is better suited for specific scientific purposes whereas the latter is much more versatile. This helps explaining how can we step outside logic even though we are in the logocentric predicament. We are in the logocentric predicament since we are bound to take the basic laws to be true. But we step outside logic whenever we reason using natural language rather than a formal language. Thus, Frege is stepping outside logic every time he writes in German. For instance, when he introduces basic law I he steps outside logic to explain, in German, that the law could not be false.

This interpretation takes at face value the controversial passage without attributing to Frege a very cryptic view. Frege thought that, due to the expressive limitations of his *Begriffsschrift,* it is sometimes necessary to formulate his views informally. This is all we

need to illuminate the metaphor of stepping outside logic. We hardly need an extra interpretative layer where Frege *shows* that negated basic laws are meaningless even though he lacks vocabulary to *say* that.

Let us consider now, how this view is analogous to the one I attributed to Kant above. I suggest that Frege took the negation of laws of logic to be inconceivable for ideal humans. That is, it is impossible for ideal humans to reject a law of thought. Of course, an actual human may fail to realize that something she asserts entails the negation of a law. Indeed, this is what happened to Frege when he thought that basic law V was true. But if we set aside the limits of actual humans, basic laws are undeniable. This, I suggest, is the "impossibility" on line eight.

On lines three to six, Frege considers an argument appealing to this impossibility to justify the laws. Then he expresses scepticism about this argument for two reasons. First, the argument shows only that ideal humans must take basic laws to be true, it hardly shows that they are in fact true (lines seven and eight). Second, we cannot exclude the possibility of logical aliens (line nine).

This should certainly give us pause. As we saw, Frege introduces basic law I by pointing out that it cannot be false. But then, in the controversial passage he seems to make the weaker claim that we cannot *take it* to be false. This creates a tension. If Frege had a stronger argument showing that basic laws cannot be false, why would he refrain from using it in the controversial passage? And, on the other hand, if he lacked such a strong argument—as the controversial passage suggests—how could he introduce the basic laws by claiming that they cannot be false?

To resolve this tension, we need to observe that basic law I cannot be false *given how Frege defined the notions in it*. This is the sort of claim that Frege is making as he introduces it—where indeed, he appeals to the stipulations he made in §12 concerning the conditional stroke. But surely under different stipulations the same formulas could be false.[60]

Frege justifies his stipulations by referring to his theory of reference. There, he never suggests that any natural language meets the standards that he sets for his formal language.[61] Rather he argues that when language is used for scientific purposes it must meet these standards—a claim which motivates his stipulations. Now, the evidence supporting Frege's understanding of scientific language seems to be his own familiarity with scientific languages and the human mind. He takes for granted claims that, according to him, reflect uncontroversial facts about scientific discourse. This makes the justification of basic laws somewhat human-centred. The laws cannot be false *in the Begriffsschrift*, which is an ideal language for human scientific discourse. But a different community might not accept the same claims about scientific language that we accept—for instance, a community might not accept that in scientific language no term should fail to refer.

Frege repeatedly envisions the possibility of mathematical aliens. He seems to take this as a nomological possibility, since he considers scenarios in which, as a result of evolution,

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[60] For instance, Frege postulates that no term in his formal language fails to refer (1893,§5). Should we remove this assumption, the truth-table of the conditional stroke would change –it would have to account not only for the cases in which antecedent and consequent are true or false, but also for the cases in which they fail to refer. This could affect basic law I since Frege appeals to the definition of the conditional stroke to justify it.

[61] For instance, he admits that natural language terms might fail to refer (1892, 40).

humans could come to believe that 2x2=3 or 2x2=5 (1884, xviii-xix; 1906e, 175). He always emphasises that arithmetical truths, and logical, are ontologically independent from human beliefs. But here he is asking how we can justify our mathematical and logical beliefs. Having agreed that some unspecified truths of arithmetic are mind-independent, the question remains open as to whether we are right to think that, say, the sentence '2x2=4' corresponds to a truth of arithmetic—rather than '2x2=3.' Frege attempts to show that this sentence must be true in a language that meets our basic scientific standards. But he also admits that this only proves that we must take it to be true. It hardly proves that '2x2=4' in fact corresponds to a mathematical truth, or that logical aliens are impossible. This echoes Kant's reasoning when he envisions the possibility that objects given to ideal humans might not be like things in themselves, and he considers the possibility of intuition aliens. And this is where Kant and Frege take different roads.

As we saw above, the claim that ideal humans must assert laws that quantify over any object whatsoever, paired with the rejection of skepticism, leads to the conclusion that said laws are true. Initially, Kant seems to take for granted that ideal humans possess some knowledge about the mind-independent world. But later he decided that this requires further argument. Thus, he wrote his "refutation of idealism" by which he was never quite satisfied.

In the controversial passage, Frege's attitude is similar to the attitude of the earlier Kant. On lines ten and eleven, Frege observes that although we cannot exclude the possibility of logical aliens, we are unable to suppose that they might be right. This is a consequence of the weak normativity of descriptive laws. If one accepts a law, one thereby accepts that everybody ought to accept it (lines fifteen and sixteen). Therefore, since ideal humans are

bound to assert laws of logic, we—non-ideal humans—may conclude that logical aliens believe something that they ought not to.[62] This is still human-centric. By the reasoning above, we cannot prove that logical aliens are wrong, but we are bound to take them to be wrong. And, conversely, *they* believe that we are mistaken. We are both stuck in our epistemic standpoints.

Now, we might be unable to rule out the possibility that our epistemic standpoint is grossly mistaken. But, says Frege speaking for himself now, questioning our epistemic standpoint is unadvisable; it would be as unadvisable as to try jumping out of one's skin (lines thirteen and fourteen). If we are to do science, we better assume that there is nothing preventing us in principle from accumulating knowledge. Rather than attempting a refutation of skepticism of the sort that Kant tried to produce time and time again, Frege suggests disregarding skepticism.

One could argue that this solution fall shorts of a justification of logic. Indeed, this is not the sort of proof that Kant hoped to give in his "refutation of idealism." But this is so because Frege is much more aware that some arguments can only be given after 'stepping outside logic.' As far as informal arguments go, Frege's reasoning in the controversial passage is sound. Skepticism is a scientific dead end. Therefore, to engage in scientific investigation, we must set it aside. If others dare doing otherwise, says Frege, one can only urgently warn them. This is not a proof, but it gives us a strong motivation to accept Frege's basic laws.

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[62] Unlike ideal humans, *we* can mistake a contradiction for a law. That is, we might assert something that we would realize we ought not to, if we reasoned unimpeded by psychological limitations. Yet, *we* know that logical aliens are wrong because, by definition, they reject the laws that ideal humans accept—whatever these laws might turn out to be.

Either we accept them, or we claim that there is no way humans can in principle accumulate knowledge.[63]

In the next section I argue that Wittgenstein, in his own framework, offered the same solution to the secondary problem of pure reason. This is hidden by the fact that Wittgenstein holds views about meaning radically different from Frege's, and Kant's. Wittgenstein claims that that "propositions of logic are tautologies" (TLP, 6.1) and "tautologies […] lack a sense" (TLP, 6.461). This suggests that he held Kant's view that the laws of logic are conditions of possibility for thought. However, unlike Kant, Wittgenstein thought that tautologies are true—not merely "a negative condition for truth" (A59/B84). Tautologies are senseless, but they are no nonsense. I discuss how these notions map to Kant's and Frege's notions to compare Wittgenstein's solution to theirs.

4. Wittgenstein's Solution

Doing TLP exegesis is a particularly difficult business because, as Wittgenstein put it in a letter to Russell: "[it is] difficult […] for me to write on logic. That's […] the reason why my book is so short, and consequently so obscure" (WC, 98).[64] Wittgenstein's remarks are notoriously open to multiple readings, or at any rate, equally competent commentators

[63] See also Frege 1918 where again Frege seems to treat entailing the actuality of a skeptical scenario as a reductio ad absurdum. He argues that a view he wants to deny entails skepticism and on that ground he discards it– the view being that mind-dependent ideas are the only possible objects of human awareness (363-8). Notably, the final piece of evidence that Frege gives before moving on is that, if we take seriously skepticism, science collapses: "the natural sciences too could only be assessed as fables like astrology and alchemy" (368).

[64] See also LO, p.19.

disagree on how to read TLP much more radically than they do on other subjects. As a result, I need to begin my discussion by listing some of the topics that I do not discuss. That being done, I draw a comparison between Frege's and Wittgenstein's terminologies and outline what Wittgenstein meant in the proposition we started from "what makes logic a priori is the *impossibility* of illogical thought" (TLP, 5.4731).

Wittgenstein's Route to the Secondary Problem of Pure Reason, and 'The Frame'

According to Hacker (1986): "Wittgenstein repudiated logicism [propositions of pure mathematics] are, strictly and technically speaking, nonsense (unlike tautologies)" (63). In this interpretation, one might say, Wittgenstein's view on the justification of logic is not an answer to the secondary problem of pure reason, because arithmetical knowledge does not reduce to knowledge about logic. Conant (1992), by contrast, holds that Wittgenstein's view improves Frege's—which, in his interpretation is self-defeating. Logical alien thought is impossible, since "we cannot make mistakes in logic" (TLP, 5.473). But the view that we cannot make mistakes in logic is itself nonsense. Thus, there is no solution to the secondary problem of pure reason because there is no problem. Here I challenge neither the idea that the propositions of mathematics are nonsense, nor the resolute interpretation. The reading I propose here concerns exclusively Wittgenstein's approach to the secondary problem of pure reason.

Wittgenstein might have thought that the propositions of pure mathematics are nonsense. But he also stated that "Mathematics is a method of logic" (TLP 6.234) and that "The logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations by mathematics" (TLP 6.22). Indeed, in his diaries he wrote:

> Light on Kant's question "How is pure mathematics possible?" through the theory of tautologies. (NB, p.15)

Clearly, Wittgenstein repudiated logicism in the sense that he rejects Frege's and Russell's approach to showing that mathematics is analytic: "the theory of classes is completely superfluous in mathematics" (TLP 6.031). And he might have gone a step further and held that the propositions of mathematics are nonsense (TLP 6.2). Yet these quotes illustrate that he was reasoning about Kant's "general problem of pure reason" (B19-B20) and that he thought that his views about logic would provide a solution to it. This is Wittgenstein's route to the secondary problem of pure reason regardless of whether he thought that the propositions of mathematics are nonsense. The task is the same as Frege's—although of course, Frege did not think that equations are nonsense—in order to provide a justification to pure mathematics, Wittgenstein gives a view on the justification of logic.

As for the resolute interpretation: what I am setting aside here is a detailed discussion about what Diamond and Conant (2004) call 'the frame' of TLP. According to the resolute reading, the main body of the book is entirely nonsense. Only some remarks—the frame—should be taken at face value, namely the remarks telling us that the theory presented in the main body is "plain nonsense" (Diamond 1991, 23) or a "mock theory" (Conant 1992, 98). Since I am setting aside the frame, for the time being, the present analysis focuses only on the allegedly nonsensical theory. Regardless of whether the frame is evidence that Wittgenstein regarded the main body of TLP as nonsense, here I assess whether Wittgenstein's theory itself—specifically the solution to the secondary problem of pure reason—support the

resolute interpretation.[65] My evaluation is that it does not. But let us not get ahead of ourselves. First, I need to compare Wittgenstein's and Frege's terminologies.

Logic Takes Care of Itself

I suggest that Wittgenstein's notion of a pseudo-propositions [*scheinsatz*]. maps onto Frege's notion of grammatically ill-formed expression (Frege 1892, 28). These are sentences—Frege would say—that cannot be *grasped* because no thought corresponds to them. On the other hand, Wittgenstein's notion of senseless tautologies maps onto Frege's notion of laws of logic and their consequences. These are sentences—Frege would say—that cannot be *rejected.*

As we saw, the notions of *grasping* and *asserting* in the *Begriffsschrift* are marked by the horizontal and judgement strokes. Now, Wittgenstein held that "Frege's judgement stroke "⊢" is logically quite *meaningless*" (TLP, 4.442). Hacker's (1986) interpretation here—with which I concur—is that Wittgenstein is making the familiar point that a valid argument can have false premises (31). What Wittgenstein does not seem to appreciate is that Frege was aware of this. Indeed, he wrote:

> The task of logic is to set up laws according to which a judgement is justified by others, irrespective of whether these are themselves true (Frege 1906e, 175).

Wittgenstein's reason to discard the judgement stroke was that, in his understanding, this symbol represents a psychological state of the authors (see NB, 96). This might be a correct

[65] Another issue I set aside is whether Wittgenstein changed his views, and how many times. See Stern 2006 for a discussion about that, among others.

assessment of Whitehead and Russell's assertion-sign,[66] but it is not a faithful depiction of Frege's judgement stroke. As we saw above, it is quite implausible that Frege held the self-evidence view. Therefore, it is unlikely that Frege meant the judgement stroke to mark his psychological attitude towards the basic laws, or any of the theorems following from them. And neither does Frege mark every step in his proofs with the judgement stroke because he thought that "one can draw inferences only from true assertions" (Hacker 1986, 31). Rather, premises in Frege's proof must be marked by the judgement stroke because, in the context of giving a foundation of arithmetic, he is concerned solely with sound arguments.[67]

Just as Frege did, Wittgenstein too distinguishes sound arguments from unsound ones. He denotes as 'tautologies' combinations of signs that are unconditionally true (TLP 4.46) and he outlines a method to recognize a given combination of signs as tautological (TLP 6.1203).

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[66] Indeed, Whitehead and Russell write:

> The sign "⊢," called the "assertion-sign," means that what follows is asserted […] For example, if "⊢(p⊃p)" occurs, it is to be taken as a complete assertion convicting the authors of error unless the proposition "p⊃p" is true. (Russell and Whitehead 1910, 8)

It is possible that Wittgenstein had this in mind when he wrote the aforementioned remarks in Costello's Notes on Logic; since there he talks about the 'assertion-sign.' But then clearly in TLP 4.442 the main critical target is Frege; since there Wittgenstein talks about the 'judgement-stroke.'

[67] Another proposition where Wittgenstein seems to misunderstand Frege is TLP 6.1271. Wittgenstein points out that all tautologies have equal status: there are no essentially primitive or derivative proposition (TLP 6.127). He then continues critically pointing out that Frege's choice of logical axioms is arbitrary: an equivalent system could be constructed with other axioms. Frege was quite aware of this (Frege 1914, 205-6).

This method amounts to semantic truth-table proofs.[68] But Wittgenstein also allows the use of other proof systems in cases where drawing a truth-table would be cumbersome. This method amounts to a syntactic proof of the sort that Frege and Russell use, since it applies rules that deal only with signs (TLP 6.126). Now, according to Wittgenstein syntactic proofs are "merely a mechanical expedient to facilitate the recognition of tautologies in complicated cases" (TLP 6.1262). This effectively takes for granted soundness and completeness results. Therefore, Wittgenstein himself postulates that his notion of tautology corresponds to Frege's axioms and theorems.

The mapping between Frege's and Wittgenstein's terminologies is imperfect because they have different views about sense. First, the bearers of sense differ. For Frege, a sentence may or may not be grammatically well-formed. If it is ill-formed, then it is nonsense. Otherwise, it has a sense, which is the way it displays its truth-value. Wittgenstein agrees that a string of symbols can violate the rules of logical syntax, resulting in pseudo-propositions.[69] Wittgenstein's propositions, however, do not correspond to Frege's well-

[68] Talking in terms of truth-tables is anachronistic. Rather than saying that the truth-table of tautologies turn out to be true for any input, Wittgenstein says that their truth-grounds are all the truth-possibilities of their truth-arguments (TLP 5.101). But using the original terms would give no clear benefit here; the contemporary notions seem to map nicely onto Wittgenstein's.

[69] No string of signs in and by itself is a pseudo-proposition. But once a proposition is assigned to a string of signs, rules of formations follow (TLP 3.342). Therefore, when we assign a proposition to a sign, other signs on that convention necessarily turn out to be pseudo-propositions. The convention can be changed giving sense to signs that formerly were pseudo-propositions, but there will again be other pseudo-propositions, and the underlying logical facts are unaffected by the choice of notation.

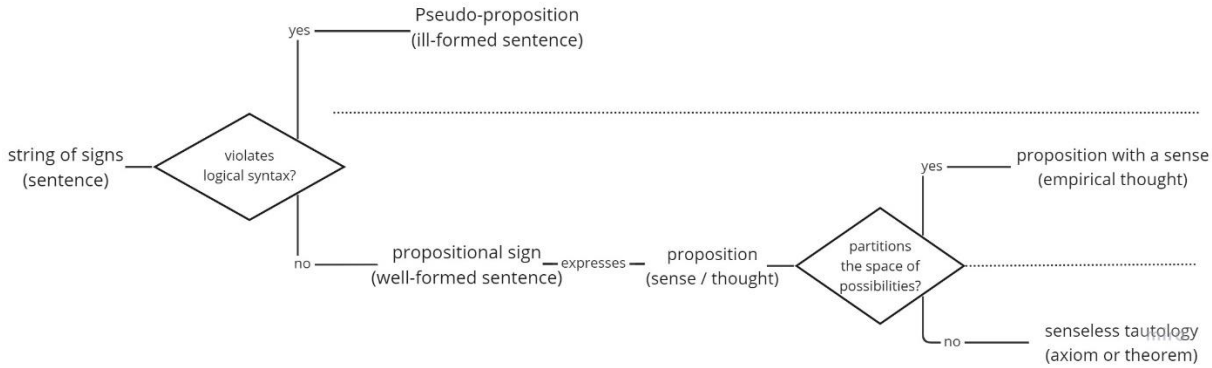
formed sentences.[70] According to Frege sentences themselves are the bearers of sense, Wittgenstein has an intermediary notion: sentences correspond to propositions, and propositions are the bearer of sense. Second, for Frege the sense of a well-formed sentence is the distinctive way it displays its reference. For Wittgenstein, a proposition's sense—if it has one[71]—is the distinctive way it partitions the space of possibilities.

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[70] Wittgenstein suggests this analogy in an early note for TLP (NB, 2), possibly because—as Anscombe suggests (NB, 15 translator's footnote)—at that time he had still in mind Frege's theory of sense. But in TLP a proposition is not a combination of signs. A strings of symbols in TLP is called "propositional sign" (TLP 3.5). A proposition is "a model of reality as we imagine it" (TLP 4.01). Commentators seem to agree on this (Conant 1992, p.64 and Hacker 2020, p.18, among others).

[71] There is some evidence suggesting that in TLP all propositions have a sense. Wittgenstein says that a picture is a model of reality (TLP 2.12). A proposition is a model of reality (TLP 4.01). Thus, a proposition is a picture. A logical picture is a thought (TLP 3). Thus, a proposition is a thought—since "every picture is *also* a logical picture" (TLP 2.182). But a thought is a proposition with a sense (TLP 4). Therefore, a proposition is a proposition with a sense. This suggests that 'sense' is a redundant notion in proposition 4; it is part of the nature of propositions that they have sense. Now if this were correct, then every time Wittgenstein talks in terms of 'propositions of logic' he would be making a category mistake, because tautologies lack a sense. Furthermore, if this were correct, proposition 4—which was one of the first focus of Wittgenstein's attention in the writing of the Prototractatus—would state a needless redundancy. This seems highly unlikely. Accordingly, I suggest that Wittgenstein did not realize that the definitions above seem to jointly entail that all propositions have a sense. Propositions may or may not have a sense. Propositions with a sense are thoughts, and they are logical pictures. Propositions without a sense are tautologies or contradictions, they picture nothing, they say nothing, but they show the logical structure of the world.

Even though the mapping is imperfect, it is possible to find a correspondence between Frege's and Wittgenstein's notions. Grammatically ill-formed sentences and pseudo-propositions correspond nicely.[72] For Frege a thought is what is expressed by a well-formed sentence; this extensionally corresponds to Wittgenstein's propositions. Indeed, according to both, there is only one sort of entity that can be expressed by well-formed sentences. Frege holds that well-formed sentences refer to truth values, but they refer to truth-values through their senses; thus, truth values are not a Fregean notion whose extension could overlap Wittgenstein's notion of proposition. For Wittgenstein a proposition has sense if it partitions the space of possibilities, this is an empirical thought for Frege. Finally, propositions lack sense when they are unconditionally true or false, these for Frege, are the thoughts expressed by laws of thought, their consequence, and their negations.



[72] Wittgenstein, unlike Frege, emphasizes that what is nonsensical in a notational convention may not be nonsensical in another. But this is hardly a point of departure from Frege. Of course, Frege was aware that a well-formed sentence in his *Begriffsschrift* need not be a well-formed sentence in, say, the language of *Principia Mathematica*. Thus, for both well-formedness is relative to a given notational convention.

This sums up the above discussion, with Frege's terms in parenthesis. I am not suggesting that the disagreement between Frege and Wittgenstein concerning the nature of sense was verbal. Although their taxonomies are extensionally equivalent, they differ intensionally. But extensional equivalence is sufficient for my purposes.

If the propositions of mathematics are tautological, and tautologies are unconditionally true, then contradictions—their negation—are inconceivable in the sense outlined above. It might seem that he thinks that tautologies and contradictions are impossible to grasp because he holds that they are senseless. But Wittgenstein's term that really corresponds to Frege's 'ill-formed sentence' is 'pseudo-proposition'—these are the strings of signs (sentences) that both take to be impossible to grasp. A tautology's senselessness amounts to its failure to partition the space of possibility—its failure to be a model of reality. But tautologies can be grasped and asserted in Frege's sense. Indeed, Wittgenstein asserts them when he says that they are "unconditionally true" (TLP, 4.461) and by recognizing that they can be proved—taking for granted soundness and completeness results.

Conant (1994) attributes to Wittgenstein the self-defeating view that contradictions cannot be grasped because he attributes to him Frege's view about sense: there is only one way that a sentence may lack a sense, which makes it neither true nor false. But Wittgenstein distinguishes pseudo-propositions, which are neither true nor false, from senseless tautologies which are most emphatically true. The graph above illustrates this: according to Frege well-formed sentences have a sense, according to Wittgenstein propositions with a sense have sense. If we superimpose Frege's terminology onto Wittgenstein's the claim that "tautologies lack a sense" entails that they are pseudo-propositions. But in Wittgenstein's

terminology it entails that they are senseless proposition which cannot be rejected.[73] This leads us back to the proposition I started with

> What makes logic a priori is the *impossibility* of illogical thought (TLP, 5.4731)

There seem to be two problems with this. First, it would seem perfectly possible for a subject to err in logic. Second, it is unclear how can we bridge the gap between inconceivability and justification. I suggest that Wittgenstein answers the first problem in the same way that Kant and Frege did: a limited human being can err in logic, but an ideal subject cannot. When Wittgenstein says that to err in logic is impossible he is talking about ideal subjects, whilst acknowledging that limited humans can make logical mistakes due to their psychological limitations. As for the second problem, it seems that Wittgenstein gave his own answer. But let us proceed in order.

Wittgenstein acknowledges that, in complicated cases, one might be unable to realize that a given sign corresponds to a tautology (TLP 6.1262). Indeed, one can learn something about the logical structure of the world by realizing that a given sign corresponds to a tautology. But once a tautology has been grasped, it is impossible to reject it. According to Wittgenstein, to think out a proposition amounts to visualizing its truth-table. An ideal agent can never fail to visualize the truth-table of a sentence, we can. But even limited humans, once they have thought out a tautology (i.e. visualised its truth-table), can never reject it. This is why 5.4731 does not mention ideal agents. What it says applies to us as well. Even non-ideal agents cannot *think* illogically, but we can fail to visualize the truth-table of a

[73] To infer that tautologies ought to be asserted since they cannot be rejected one needs now-controversial background assumptions on the nature of negation. See footnote 55.

sentence. It is possible to think illogically in the sense that one can mistakenly associate a false proposition to a propositional sign that is tautological in the notation in use. However, it is impossible to think illogically in the sense that one can never reject a tautological propositional sign that one has applied and thought out (i.e. having visualized its truth-table).[74]

It is often remarked that, unlike Frege and Russell, Wittgenstein thought that ordinary language is in perfect logical order (Anscombe 1959, 91-2). The *Begriffsschrift* is a technical tool developed to replace ordinary language for specific scientific purposes, analogous to a microscope. Wittgenstein's conceptual notation, by contrast, is meant to make evident the underlying logical structure of ordinary language. Accordingly, Frege needs to explain why one should adopt his tool rather than some other; his answer is that in the *Begriffsschrift* the laws of thought, which ideal humans cannot reject, must be true. Wittgenstein does not face this problem because in his view we already use a language in which tautologies cannot be rejected—only we often fail to realize that we do. It is in this sense that logic takes care of itself.

Again, we can clarify this by considering the difference between Frege's and Wittgenstein's terminology. Wittgenstein says that language is the totality of propositions (TLP 4.001). That is, language is the totality of what propositional signs express. Frege, by contrast, talks about languages as collections of symbols and rules to manipulate them. For instance, Frege (1893) says that the expression 'there is' in ordinary language is rendered in his

[74] This is factually false, non-classical logicians such as Priest (1987) reject classical tautologies and it is absurd to say that they fail to visualize their truth tables. Of course, Kant, Frege and Wittgenstein had never encountered logicians seriously defending non-classical logics at the time of writing.

Begriffsschrift by the concavity of the horizontal stroke (§13, referring to §8). This suggests that his formal language is intended as a symbolism that can express more precisely the same thoughts that ordinary language can express. For Wittgenstein, on the other hand, language is the collection of thoughts expressed by various notations. [75]

Frege could very well agree that language is in perfect logical order if language is understood as the totality of what he called thoughts. Indeed, I argued that he took the laws of thought to be impossible to reject for ideal humans. If this is right, then ideal humans can never err in logic, since they can never fail to associate the correct thought to a sentence and they can see all syntactic consequences of any thought they grasp. Wittgenstein's conception is intensionally different because he takes a semantic approach to logic. Ideal humans can never err in logic because they can always immediately visualize the truth-table of any sentence. Ordinary humans can never err in thought since to think through a

[75] Wittgenstein's use of the term 'language' alternates these two meanings. In TLP 3.325, for instance, he uses interchangeably 'sign-language' [*zeichensprache*], 'conceptual notation' [*begriffsschrift*], and 'language' [*sprache*]. This ambiguity is plausibly what led to the disagreement with Russell. Russell thought that the Tractatus aims to describe a logically perfect language (see Russell's introduction to TLP). This reading is quite correct if a language is a notational convention, indeed Wittgenstein among other things develops a notational convention in the Tractatus. It is absolutely wrong if language is understood as the collections of thoughts to be expressed by our notational conventions. Our everyday notational convention is in perfect logical order since it expresses the same thoughts that any notation does—its problem is that it disguises their logical structure. See also WC, 98.

sentence amounts to visualizing its truth-table—but we can err in the sense that we may fail to recognize a tautology in complicated cases.

This is in essence the same view that Kant and Frege held. Our logical or intuitive errors result from psychological limitations. If we had no psychological limitations, those errors would be impossible. Concerning the second point, I suggested above that Kant and Frege intended to bridge the gap between inconceivability and justification by rejecting Cartesian skepticism. In the second edition of the first critique Kant attempted an explicit refutation of skepticism—one that never quite satisfied him. In the first edition of the first critique, and in the *Grundgesetze,* skepticism is simply set aside.

This move is available for Wittgenstein as well. Indeed, he wrote "scepticism is not irrefutable but obviously nonsensical" (TLP, 6.51). But he does not seem to have linked this with the justification of logic. Moreover, while Kant had recognized the possibility of intuition aliens and Frege the possibility of logical aliens, Wittgenstein never seems to accept the possibility of logical aliens. On the contrary, all his remarks about illogical thought emphasise its impossibility (TLP, 3.03, 3.031, 5.4731). In fact, in Wittgenstein's case we need not talk in terms of ideal *humans*. His focus is language itself, not any of the subjects that express it. Now, as to the reason why the structure of language should correspond to the structure of the world Wittgenstein writes.

> *The limits of my language* mean the limits of my world (TLP, 5.6)
>
> Logic pervades the world: the limits of the world are also its limits (TLP, 5.61)

As we saw, there are two senses in which language is said to have limits in the Tractatus. On the one hand language is limited because nonsense cannot be expressed; as soon as an

arbitrary string of signs is assigned to a proposition, other strings of signs turn out to be pseudo-propositions. On the other hand, language is limited because tautologies cannot be rejected; one can reject a tautological sign only if one fails to grasp its truth-table. Now, the limits of thought, says TLP 5.6, coincide with the limits of the world. Therefore, the investigation carried out in the Tractatus amounts to a study of the world itself.

It is a complex exegetical problem in its own right—one that I do not tackle here—to evaluate Wittgenstein's notes about solipsism. For the moment it suffices to say that TLP 5.6 is the proposition most clearly stating that the gap between inconceivability and justification can be filled. Thus, whatever reason Wittgenstein had in support for it, that presumably was his main reason to think that logic has a bearing on reality. Hacker (1986) argues that Wittgenstein was influenced here by Schopenhauer's view that "the world is my representation" (1818, 3). If this is correct, then although Wittgenstein's answer to the secondary problem of pure reason has much in common with Kant's and Frege's, ultimately, they appeal to radically different devices. Kant and Frege reject Cartesian skepticism on the grounds that it is a scientific non-starter: if we are to do science, we need to presuppose that, at least ideally, humans can obtain knowledge. Wittgenstein, by contrast, appeals to Schopenhauer's notion of a non-psychological "metaphysical subject" that constitutes the limits of the world by representing it.

5. Conclusion

We started observing that, having claimed that our knowledge of arithmetic reduces to a second sort of knowledge, the question arises as to how to justify the second sort of knowledge. Kant, Frege, and Wittgenstein approach this problem from different angles. Their secondary targets are different—intuition for Kant and logic for Frege and

Wittgenstein—and their methodologies are different. Nevertheless, their solutions are analogous: they all appeal to some sort of inconceivability: the boundaries of our ideal cognition, they claim, ground the justification of the basic elements in their philosophical systems. In this interpretation, the precursor of Frege's solution are Kant's views on intuition, not logic. Frege's system is only self-defeating in the familiar sense. And the Tractatus is not meant to be plain nonsense. The core idea is that the inconceivability at play in Kant, Frege, and Wittgenstein amounts to an ideal impossibility to *reject,* rather than *grasp*, the basic elements of their philosophical systems.

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