

LIBERAL ARTS GEOMETRY

Viktor Bläsjö

© 2019 intellectualmathematics.com

CONTENTS

0	Syllabus	1
1	Beginnings of mathematics	2
2	Elements 47–48: Pythagorean Theorem	3
3	Textual aspects of Greek mathematics	5
4	Elements 34–41: parallelograms; area	7
5	Beginnings of geometry	8
6	Elements Defs.–Prop. 3: foundations; ruler and compass	12
7	Plato and Aristotle on geometry	13
8	Elements 4–11: triangle congruence; constructions	15
9	Ruler and compass	16
10	Constructivism	17
11	Mathematics and culture in early modern Europe	22
12	Perspective art	27
13	Projective geometry	31
14	Conic sections	34
15	Geometry on surfaces	36
16	450 degree cone	37
17	Elements 13–26: angles and congruence	39
18	Early modern reflections on mathematical method	40
19	Elements 27–32, 46: parallels	44
20	Rationalism versus empiricism	44
21	Abstract axiomatics	50
22	Hyperbolic geometry	51
23	Space and perception	52
24	Abstract versus physical geometry	56
25	Polyhedra	58
26	Cultural history of trigonometry	60
27	General discussion questions	60

§ 0. Syllabus

Content. In this course we take a big-picture look at the nature of mathematics and its role in human thought, emphasising its interactions with society, history, philosophy, science, and culture. The course involves studying key mathematical arguments in some detail, but our goal is not to develop a repertoire of technical skills. Instead, we study accessible mathematical topics specifically selected for their rich interconnections with cultural context, and integrate our mathematical work with reflections on its broader meaning and implications. Instead of the drill and practice problems of a traditional mathematics class, we approach mathematics through seminar discussions, hands-on activities, and readings connecting it to broader issues. We thus analyse a selection of emblematic and important mathematical proofs and use them as a platform for reflecting on the nature of mathematics. In parallel, we read excerpts from seminal historical texts across the ages as well as modern scholarship from a wide range of academic disciplines that shed light on the interplay between mathematics and its

societal and intellectual context. We focus especially on geometry, from the origins of mathematical reasoning in early civilizations, to Euclid's *Elements* that was the gold standard of exact reasoning for millennia and the model for countless philosophical systems, to the projective geometry of Renaissance art, to the more modern non-Euclidean geometry that overturned conventional wisdom about the nature of human spatial perception and the shape of space.

Aims. After completing the course students are able to:

- Discuss critically, and situate in scholarly and historical context, reflections on the nature of mathematics.
- Draw on a range of examples to discuss the role of mathematics in society, culture, and human thought.
- Read simple mathematical proofs and explain the ideas and methodologies involved.
- Reason rigorously and conceptually about fundamental notions in a simple geometric setting.
- Relate fundamental aspects of mathematical reasoning to its broader meaning and purpose.
- Discuss the nature of mathematical knowledge and axiomatic-deductive systems, especially on the basis of key foundational concepts of Euclidean and non-Euclidean geometry.

Format. We interleave mathematical topics with seminar discussions of a rich array of short readings that connect the material to a broader cultural, philosophical, and historical context. We study a selection of mathematical proofs and topics drawn from geometry with an emphasis on conceptual understanding. To this end we supplement textual mathematical sources with physical models and hands-on activities that allow us to experience and explore geometry in a concrete way.

Class preparation. The course involves three types of classes:

- Seminar classes. These are the sections that contain reading excerpts from various sources and associated discussion questions marked $\mathcal{Q}$. For these classes it is essential to have read the text in advance and come prepared to contribute to a discussion.
- Activity classes. These are the sections that outline a mathematical idea and sets up some mathematical problems or investigations. For these sections, look at the text before class and read the “story” part of it; the problems and activities we will then tackle together in class.
- Euclid classes. These are the sections whose title starts with “Elements.” For these classes you should read the indicated parts of Euclid's *Elements* carefully before class. We will then discuss the associated problems and discussion questions in class.

Assessment.

- 25% Midterm exam
- 25% Final exam

Midterm and Final exams will consist of:

- 4 questions based on our seminar reading and discussions. I will give you a list of 7 questions selected from the seminar discussion questions marked $\circ$. Out of these 7 questions, you answer 4 of your choosing.
- One question asking you to prove a particular proposition from Euclid. This will be one of the following propositions:

* Midterm: 2, 5, 9, 11

* Final: 13, 32, 35, 46

You will provide a step-by-step proof with a figure and a justification for each step. Justifications can refer to previous propositions, postulates, and definitions from Euclid. At the exam, you will be allowed to use the single-page reference table of Euclidean propositions.

40% Assignments

Assignments are the questions recognisable by the $\square$ symbol. Most of these questions we will investigate together during Euclid and Activity classes. As we figure them out, you will take notes and later submit your answers.

10% Participation

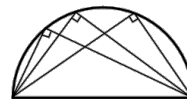
Participation means active engagement in discussions. Criteria for an A grade:

- For each Seminar class, you come prepared with a question or comment that you think could make for interesting discussion, in addition to at least one good bullet point in reply to each of the assigned discussion questions.
- For each Euclid class, you have studied the assigned proofs, and you have looked at the assigned discussion questions and have at least an idea or question to get you started toward collectively resolving the questions in class.
- In discussions, you make substantive connections across different readings and materials in the course, or with other relevant background knowledge you may have.
- In discussions, you make an effort to understand the points of view of others and engage with them thoughtfully.

§ 1. Beginnings of mathematics

The Greek tradition has it that Thales (c. –600) was the first to introduce deductive reasoning in geometry. Two of the results he supposedly proved were:

- A diameter of a circle cuts the circle into two equal halves.
- Triangles raised on the diagonal of a circle all have a right angle (“Thales’ Theorem”).



1.1. How do you think Thales proved these results?

Since this was the very beginning of geometry, Thales had to prove these things from scratch, or using arguments that his audience would accept as “obviously” true. He did not use any “formulas” or any other established principles from some textbook, because there were no textbooks. This is why it is interesting to try to recreate his steps: Forget everything you have heard about geometry and do it yourself! Enjoy the freedom of mathematical exploration! Create something out of nothing!

Here are some hints if you get stuck. The first theorem is so “obvious” that it’s hard to know where to start. In such cases, a useful trick is often to use a proof by contradiction: instead of proving that the theorem is true, prove that it could not *not* be true. That is to say, assume for the sake of argument that a diameter in fact cuts a circle into two pieces that are *not* equal. Now try to prove that this is impossible. You can do this by proving that this assumption implies absurd consequences.

The second theorem probably calls for a rather different strategy. There are many ways of proving it. I will give you a hint that sets you on one fruitful path. Draw a rectangle. Get yourself a ruler and compass. How can you play or doodle with those tools, starting with the rectangle? What lines and circles feel most natural to add to the picture? Perhaps you will find yourself with a “Thales’ Theorem” configuration before you know it. What does this mean?

I hope these examples gave you a taste of discovery and exploration and mathematics “in the wild,” without the artificiality and shackles of a modern mathematics textbook. “The essence of mathematics is its freedom.”¹ This is how mathematicians think of their subject. It is to their great regret and frustration that others perceive their subject very differently. The following reading argues this point passionately.

1.2. Read pages 1–7 of “Lockhart’s Lament,” available at https://www.maa.org/external_archive/devlin/LockhartsLament.pdf. The author compares mathematics to art. Which of the following are true for both mathematics and art, according to the author?

- ☐ Schools destroy interest in it.
- ☐ Children have a natural interest in it.
- ☐ It cannot exist without creative expression.
- ☐ Most people don’t understand what it’s all about.
- ☐ Modern students are less receptive to it because of smartphones and the Internet.

¹Georg Cantor, *Mathematische Annalen*, Bd. 21, 1883, p. 564. “Das Wesen des Mathematik liegt gerade in ihrer Freiheit.”

Let's consider an interesting application of Thales' Theorem. In Virgil's *Aeneid*, we encounter a woman who appears to have been mathematically educated: Dido, daughter of the king of Tyre, a major city in antiquity, whose ruins can still be seen in present-day Lebanon.

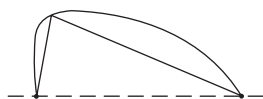


Dido was forced to flee her home and ended up on the north coast of Africa, where she struck a bargain to buy as much land as she could enclose with an oxhide.

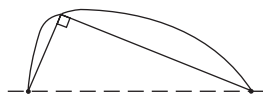


She cut the hide into thin strips, and then she faced the mathematical problem of enclosing the largest possible area within a given perimeter. Perhaps drawing on her royal mathematical education, it seems she decided to form a semi-circular perimeter next to the shoreline.

- 1.3. Show that this was a wise decision as follows. Suppose Dido had made a shape that was not a semicircle. Then by Thales' Theorem there will be some point on the boundary where lines drawn from the endpoints on the shoreline meet at an angle that is not right.



Think of there as being a void inside the triangle and think of the pieces on the sides as glued on. Slide the endpoints along the symmetry line to make the angle right.



Argue that this increases the area, and that hence a non-semicircular shape is sub-optimal.

Despite her mathematical acumen, Dido did not get to rule over her round city for very long. Aeneas, on his quest to found Rome, is shipwrecked and blown ashore at Carthage. Dido falls in love with him, but he does not return her love. He sails away and Dido kills herself.



"And so an ungrateful and unreceptive man with a rigid mind caused the loss of a potential mathematician. This was the first blow to mathematics which the Romans dealt."²

§ 2. Elements 47–48: Pythagorean Theorem

Read Euclid's *Elements*, Propositions 47–48.

Euclid's *Elements* is one of the most important and influential works in human history. We shall read enough of it to get a sense of why. "Euclid alone has looked on beauty bare,"³ as the poets say.

The first reading assignment is Euclid's proof of the Pythagorean Theorem. Soon we will go back and read the *Elements* from the beginning, but starting with the Pythagorean Theorem will help us appreciate the earlier parts. If we read Euclid from cover to cover, we get a strictly "bottom-up" perspective: we start with the most basic things and gradually get to higher and higher levels of sophistication. That is how mathematics is typically written down, and with good reason. But the way mathematics comes into being is a much more bidirectional process. Mathematics grows like a tree: as the branches extend, so do the roots. Starting our Euclid adventure with the Pythagorean Theorem is a way of making us think about this.

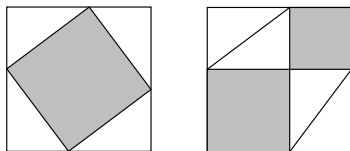
You will find when you read Euclid's proof of the Pythagorean Theorem that it is based on earlier results. We will study them in turn later, but for now it is enough to refer to the overview reference table and visual summary of the theorems of the *Elements* to know what previous results Euclid has in mind.

²Morris Kline, *Mathematics for the Nonmathematician*, Dover, New York, 1985, p. 135.

³Edna St. Vincent Millay (1892–1950).

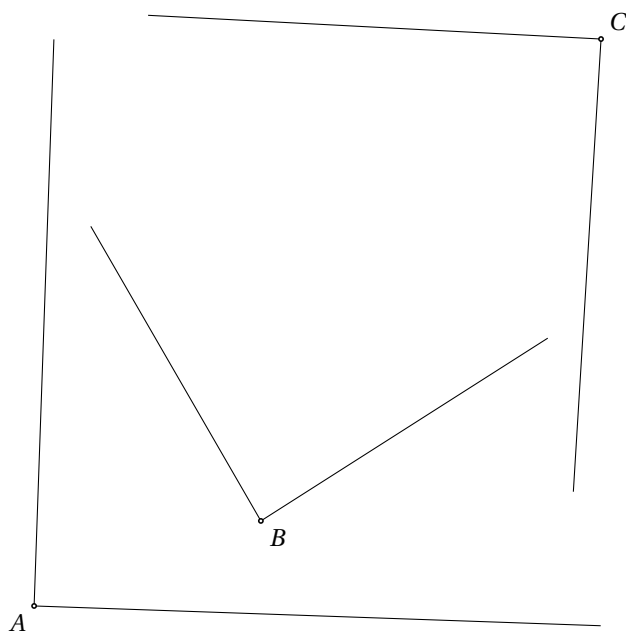
If we visualise the Pythagorean Theorem as the apex of a pyramid, the proof reveals which lower, more foundational stones it rests on. Those stones in turn rest on other stones, and so on. Something has to be the bedrock that is considered solid enough not to need any further support beneath it. Euclid's *Elements* can be read in two directions: as a way of *building up* a more and more elaborate structure on top of solid foundations, or as a way of *reducing* advanced results to their basic components. Indeed, by the time Euclid write the *Elements*, the theorems themselves—such as the Pythagorean Theorem—had been known for hundreds or even thousands of years. But it is one thing to have a bunch of theorems, and it is quite another to systematically derive all of them from an extremely restrictive set of basic assumptions. Euclid is very much concerned with the latter. So when we read the proof of the Pythagorean Theorem, one of the perspectives we should use is to think of it as “boiling down” this somewhat advanced result to more basic ones. This will help us appreciate the purpose and achievement of the more fundamental part of the *Elements* when we get to those.

- 2.1. Explain how the figure below proves the Pythagorean Theorem. Why did Euclid prefer his “windmill” proof?



- 2.2. Which is a right angle? Use Proposition 48 to find out.

- ☐ A
☐ B
☐ C



- 2.3. There are some indications that the ancient Egyptians knew that a triangle with sides 3, 4, 5 has a right angle.

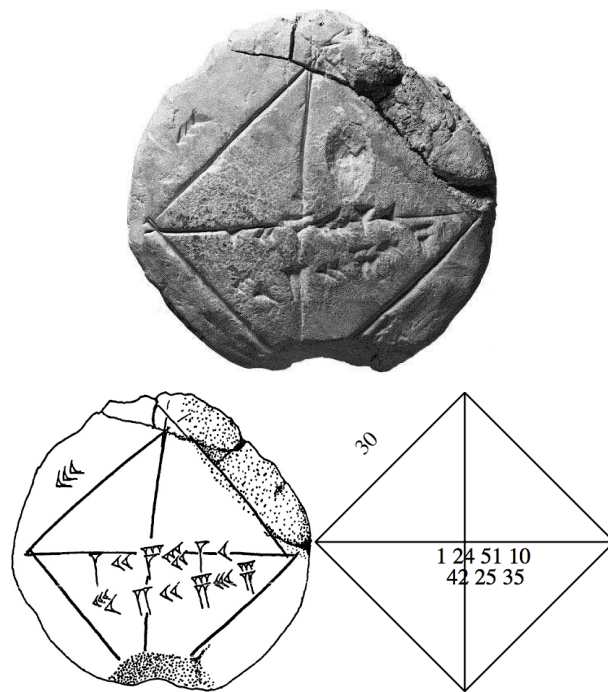
Is it realistic that they used this to construct right angles for practical purposes, such as when building the pyramids?

- ☐ Yes
☐ No

- 2.4. In <https://youtu.be/I4tK0vDKhE8>, what is presented as the most likely motivation for the discovery of the Pythagorean Theorem?

- ☐ Determination of the sizes of fields.
☐ Determination of geographical distances.
☐ Practical engineering questions such as what size ladder is needed for a certain task.
☐ Astronomical applications having to do with eclipses.
☐ No practical motivation, rather general interest in mathematical challenges/play/curiosity.

The Pythagorean Theorem has little to do with Pythagoras. It was in effect known long before. It is arguably implied in this ancient Babylonian clay tablet from almost 4000 years ago:



The numbers are written in sexagesimal (base 60) form, a Babylonian invention that still lives in our systems for measuring time and angles. This means that, for example, $42, 25, 35 = 42 + \frac{25}{60} + \frac{35}{60^2}$.

- 2.5. (a) Explain the meaning of the numbers on the tablet. Hint: there are three numbers and one of them is $\approx \sqrt{2}$.
(b) Convert the tablet's value for $\sqrt{2}$ into decimal form.
(c) If you use this value to compute the diagonal of a square of side 100 meters (i.e., roughly the size of a football field), how big is the error?

- ☐ less than 1 mm
- ☐ 1 mm – 1 cm
- ☐ 1 cm – 5 cm
- ☐ 5 cm – 10 cm
- ☐ 10 cm – 20 cm
- ☐ 20 cm – 50 cm
- ☐ more than 50 cm

§ 3. Textual aspects of Greek mathematics

- 3.1. ☞ Is the written record a good representation of Greek geometrical thought?
- 3.2. ☞ Were Greek ways of recording mathematics constrained by technology? By tradition?
- 3.3. ☞ Was Euclid a Platonist who reasoned about eternal, abstract objects, or did he think of geometry as something physically produced by ruler and compass? What can we conclude in this regard from the passive formulations of construction steps in his proofs?
- 3.4. Euclid's propositional statements are set up in a form suitable for:
 - ☐ mental visualisation
 - ☐ phonetic memorisation
 - ☐ constructive drawing

REVIEL NETZ, *The Shaping of Deduction in Greek Mathematics*, Cambridge University Press, 2003.

“THEAANDTHEBTAKENTOGETHERAREEQUALTOTHECANDTHED This is how the Greeks would write $A+B=C+D$, had they written in English. And it becomes clear that only by going beyond the written form can the reader realise the structural core of the expressions. Script must be transformed into pre-written language, and then be interpreted through the natural capacity for seeing form in language. Greek mathematical formulae are post-oral, but pre-written. They no longer rely on the aural; they do not yet rely on the layout.” (163)

“The lettered diagram is a distinctive mark of Greek mathematics. ... No other culture developed it independently.” (58) “The overwhelming rule in Greek mathematics is that propositions are individuated by their diagrams” (38), contrary to the economy of using the same diagram for several propositions, and contrary even to plain sense, it would seem, in the use of completely functionless diagrams for number-theoretic propositions (41).

But the diagrams were schematic only, with for example conic sections being crudely represented by circular arcs (34). The diagrams were also static since “of the media available to the

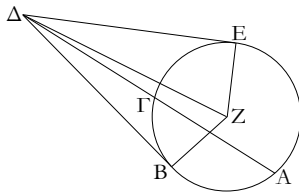
Greeks ... none had ease of writing and rewriting” (14). Standard media were papyri and wax tablets, and, for larger audiences, such as Aristotle's lectures, “the only practical option was wood ... painted white” (16). “None of these [ways of representing figures] is essentially different from a diagram as it appears in a book. ... The limitations of the media available suggest ... the preparation of the diagram prior to the communicative act—a consequence of the inability to erase.” (16) “This, in fact, is the simple explanation for the use of perfect imperatives in the references to the setting out—‘let the point A have been taken’. It reflects nothing more than the fact that, by the time one comes to discuss the diagram, it has already been drawn.” (25)

KEN SAITO, Diagrams and traces of oral teaching in Euclid's *Elements*: labels and references, *ZDM Mathematics Education*, 50(5), 2018, 921–936.

“The teaching of mathematics in ancient times was prevalently oral.” (§2.3) “The written demonstration is rather an exceptional form of transmitting mathematical ideas to a person living far away, and written text was probably not the main task of a mathematician.” (§7) Many aspects of Greek texts are explained by this circumstance, for example why propositions are never referred to by number (§3.1) or why points are labelled in alphabetical order as they are named in the proof, without any regard for continuity of notation across sequences of very closely related propositions (§3.2).

“A proposition of Euclid's *Elements* begins with so-called ‘protasis’, or general enunciation, where the proposition is stated in a general way, without a diagram. ... For example, the protasis of proposition III.13 goes: ‘A circle does not touch a circle at more than one point, whether it touches it internally or externally.’ This is fine. One understands what it purports. However, a protasis can become so long and complicated, that it is difficult to understand it. The proposition III.37 ... offers a good example. ‘If a point be taken outside a circle and from the point two straight lines fall on the circle, and if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.’ Probably few people are endowed with such intelligence as to understand without difficulty what this proposition means on reading it for the first time. So when we read Euclid's *Elements*, we often skip the protasis and begin with the ekthesis, or setting out, which follows the protasis and explains the premises of the protasis by assigning names to the points. ‘For let a point Δ be taken outside the circle $AB\Gamma$; from Δ let the two straight lines $\Delta\Gamma A$, ΔB fall on the circle $A\Gamma B$; let $\Delta\Gamma A$ cut the circle and ΔB fall on it; and let the rectangle $A\Delta$, $\Delta\Gamma$ be equal to the square on ΔB .’ Then Euclid's restates the conclusion with the names of points. This part is called diorismos (specification). ‘I say that ΔB touches

the circle $AB\Gamma$.” (§3.4)



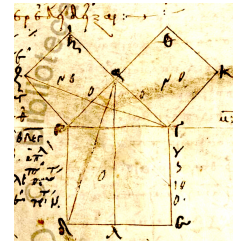
“Now, if the ekthesis, setting out, with diagram and names of points, is much easier to understand, why does the text of the *Elements* always preserve the protasis, which is often skipped by modern readers? Moreover, in many books of the *Elements*, the protasis is almost literally repeated at the end of each proposition as *sumperasma* (conclusion), adding only one word “*ara*” (therefore) to the protasis. Taking into account the difficulty and high cost of copying and preserving long text in antiquity, the generosity of the author (and/or ancient editors) of the *Elements* for protasis and *sumperasma* is quite impressive. Why did it not occur to them to reduce the length of propositions by suppressing the protasis? The answer should be that the protasis (repeated in *sumperasma*) deserved the space it occupied. But what was its value or function for ancient mathematicians and teachers?” (§3.4)

Since, in oral teaching, “the diagram of a proposition was probably erased when the next proposition had to be treated, ... there must have been some way of memorizing and referring to the propositions. I believe that this was exactly the role of the protasis. Indeed, although a protasis such as that of III.37 is long and incomprehensible to someone who reads it for the first time, it is not so hard to memorize after one has learned the proposition and one has understood what it means. ... Another important function of the protasis is that a protasis is useful when you want to apply the proposition you know in the demonstration of a later proposition. The protasis is a quotable format. ... If you want to apply a proposition whose protasis you have in memory in a demonstration you are working on, you recite the protasis, replacing the indication of geometrical objects by the expression with labels in the diagram, and you have the argument you need.” (§4.4)

CHRISTIÁN C. CARMAN, Accounting for overspecification and indifference to visual accuracy in manuscript diagrams: A tentative explanation based on transmission, *Historia Mathematica*, 45, 2018, 217–236.

“The first time you encounter a medieval manuscript of a Greek mathematical or astronomical work, like those of Archimedes, Euclid, or Aristarchus, the most impressive feature is the odd configuration many diagrams show. There is a tendency to represent more regularity among the geometric objects than what the argument demands and usually they are not accurate graphical depictions of the mathematical object discussed in the text.” (217)

Here for example is a typical manuscript depiction of the figure for the Pythagorean Theorem (*Elements* I.47):



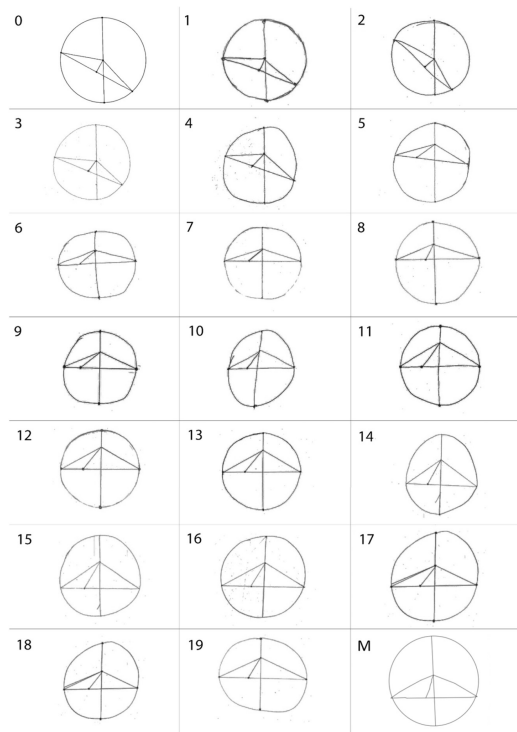
Even though the theorem holds for any right-angle triangle, the manuscript has drawn the special case where the two legs are equal, thereby giving the misleading impression that the theorem is less general than it really is.

“Considering that, except for very few testimonies extant in papyrus (from which not much can be inferred), the most ancient witnesses of Greek diagrams are medieval, we cannot conclusively decide whether these characteristics go back to the Greek authors or they are the result of medieval copyists.” (222)

One hypothesis is that, “starting from the supposed original diagram (that is, the diagram well done), a series of copies will generally transform it into a diagram similar to that of the manuscripts.” (222) It is indeed plausible that repeated copying converges to specificity, assuming copyists largely ignorant of mathematical content. For example, in the case of the Pythagorean Theorem, a scribe might get a version where the legs look similar and mistakenly assume that exact equality was intended. He then copies it this way, and specificity is introduced. No one will restore more generality in the diagram, because that would require mathematical understanding.

“To test this hypothesis I asked many groups of university students to play the role of copyists. ... I gave a folder to one of them, containing a drawing of the diagram as I presume the Greek author drew it. I asked the student to copy the diagram and to place her copy in the folder, over the starting diagram she had copied before hiding it, and to pass the folder to the next student, who would copy the drawing made by the first student, and so on.” (222–223)

Below is an example of the successive copies of the original figure (O), which do indeed approach the surviving manuscript figure (M). Note for example that the short segment is supposed to be perpendicular to the long one.



another. It was consequently necessary to measure the land repeatedly. Hence it is said geometry originated here, as the art of keeping accounts and arithmetic originated with the Phoenicians, in consequence of their commerce.” (17.1.3)

PROCLUS, *A Commentary on the First Book of Euclid's Elements*, c. 450, translated by Glenn R. Morrow, Princeton University Press, 1992.

“Geometry was first discovered by the Egyptians and originated in the remeasuring of their lands. This was necessary for them because the Nile overflows and obliterates the boundary lines between their properties. It is not surprising that the discovery of this and the other sciences had its origin in necessity, since everything in the world of generation proceeds from imperfection to perfection. Thus they would naturally pass from sense-perception to calculation and from calculation to reason. Just as among the Phoenicians the necessities of trade gave the impetus to the accurate study of number, so also among the Egyptians the invention of geometry came about from the cause mentioned.” (52)

LEONARD MLODINOW, *Euclid's window: The Story of Geometry from Parallel Lines to Hyperspace*, Touchstone, 2002.

3.5. ☞ Herodotus, Strabo, and Proclus give subtly different accounts of the origins of geometry in Egypt. Based on this information alone, are there grounds for judging their relative credibility?

HERODOTUS, *The Histories*, c. -440, translation by A. D. Godley.

“This king [Sesostris] also (they said) divided the country among all the Egyptians by giving each an equal parcel of land, and made this his source of revenue, assessing the payment of a yearly tax. And any man who was robbed by the river of part of his land could come to Sesostris and declare what had happened; then the king would send men to look into it and calculate the part by which the land was diminished, so that thereafter it should pay in proportion to the tax originally imposed. From this, in my opinion, the Greeks learned the art of measuring land.” (2.109)

STRABO, *Geographica*, c. -10, quoted from *The Geography of Strabo*, translated by H.C. Hamilton & W. Falconer, London, 1903.

“An exact and minute division of the country was required by the frequent confusion of boundaries occasioned at the time of the rise of the Nile, which takes away, adds, and alters the various shapes of the bounds, and obliterates other marks by which the property of one person is distinguished from that of

“Picture a windswept, desolate desert, the date, 2580 B.C. The architect had laid out a papyrus with the plans for your structure. His job was easy—square base, triangular faces—and, oh yeah, it has to be 480 feet high and made of solid stone blocks weighing over 2 tons each. You were charged with overseeing completion of structure. Sorry, no laser sight, no fancy surveyor's instruments at your disposal, just some wood and rope. As many homeowners know, marking the foundation of a building or the perimeter of even a simple patio using only a carpenter's square and measuring tape is a difficult task. In building this pyramid, just a degree off from true, and thousands of tons of rocks, thousands of person-years later, hundreds of feet in the air, the triangular faces of your pyramid miss, forming not an apex but a sloppy four-pointed spike. The Pharaohs, worshipped as gods, with armies who cut the phal-luses off enemy dead just to help them keep count, were not the kind of all-powerful deities you would want to present with a crooked pyramid. Applied Egyptian geometry became a well-developed subject.” (§2)

§ 4. Elements 34–41: parallelograms; area

4.1. The simplest sense in which two figures can have the same area is that of superposition: one “fits” on top of the other. But figures can also have the same area without being capable of such alignment; that is, without having the same shape. The first time Euclid talks about equality of area in this sense is in Proposition:

- ☐ 34
- ☐ 35
- ☐ 37
- ☐ 41

4.2. In Propositions 34–41, Euclid’s strategy for proving areas equal is to:

- ☐ Transform one area into another with an area-preserving transformation.
- ☐ Cut them into pieces and use triangle congruence.
- ☐ First establish area of three-sided figures, then use this to establish area of four-sided figures, and so on upwards.

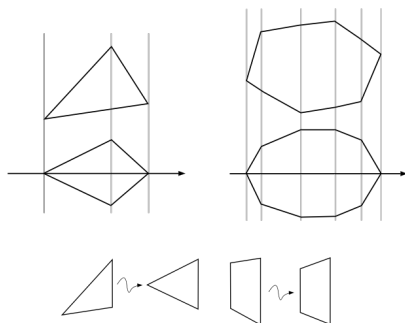
4.3. Proposition 35. If AD and EF overlap:

- ☐ Euclid’s proof can be considered to still apply, with G being above the parallelograms.
- ☐ A longer proof is needed.
- ☐ A shorter proof is possible.
- ☐ The proposition no longer holds.

4.4. The enemy is advancing toward your capital. They have currently set up camp not far from you. In the night, you send a spy to estimate the size of their army. The spy reports back that the enemy camp has the shape of a parallelogram and is 4000 paces all the way around. It is estimated that each soldier uses about 10 square paces of area for their night camp. Approximately how many soldiers does the enemy have in the camp?

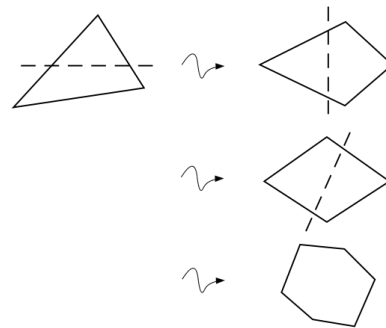
- ☐ 4000
- ☐ 40000
- ☐ 400000
- ☐ 1000
- ☐ 10000
- ☐ 100000
- ☐ Cannot be determined from the information given.

4.5. This process is called “symmetrisation”:



In particular, it turns parallelograms into rectangles, which can be likened to straightening out a stack of

books that has been knocked askew. Here is the symmetrisation process applied repeatedly to a specific figure:



The symmetrisation process is akin to packing a snowball; both these processes lead to a round shape for the same reason.

Among all figures with the same area, the circle has the least perimeter. I can prove this using symmetrisation if I first assume or prove that:

- ☐ Symmetrisation preserves area but decreases perimeter for any non-circular figure.
- ☐ Symmetrisation has no effect on circles.
- ☐ Among all figures with the same area, there exists one that has the least perimeter.

§ 5. Beginnings of geometry

- 5.1. ☞ Why did the conception of mathematics as a field of knowledge characterised first and foremost by rigorous deductive reasoning arise in ancient Greece?
- 5.2. ☞ What does early Greek philosophy have in common with mathematics? What are some differences between the two?
- 5.3. ☞ Why did the mathematicians care what the philosophers were doing?
- 5.4. ☞ In what ways can Zeno be taken as a case in point illustrating the main theses of Lloyd and Szabó?
- 5.5. ☞ Are Zeno’s two arguments just different literary elaborations of the same idea, or is there something substantially different between the two cases?
- 5.6. Why did the conception of mathematics as a field of knowledge characterised first and foremost by rigorous deductive reasoning arise in ancient Greece? The root cause, if anything, is arguably:
 - ☐ economics
 - ☐ politics
 - ☐ geography
 - ☐ interaction of cultures

5.7. Early Greek thought was:

- ☐ combative
- ☐ collaborative
- ☐ egalitarian
- ☐ reverential
- ☐ linked with practice

5.8. In ancient Greece, the group most eager to separate mathematics from physics were the:

- ☐ geometers
- ☐ philosophers
- ☐ scientists
- ☐ teachers

5.9. The axiomatic-deductive method arose due to:

- ☐ The nature of geometry specifically.
- ☐ The nature of mathematics generally.
- ☐ Philosophical critiques.
- ☐ The discovery of multiple possible geometries.
- ☐ Errors encountered in intuitive reasoning.

G. E. R. LLOYD, *Magic, Reason and Experience: Studies in the Origin and Development of Greek Science*, Cambridge University Press, 1979.

“Public debates between contending speakers in front of a lay audience” was a prominent part of ancient Greek culture (93). Science and philosophy were born on this stage. Many otherwise peculiar characteristics of Greek thought are explained by this format.

The stage debate requires the speakers to proclaim bold and provocative theses, and to strive to avoid reconciliation with other viewpoints at all costs. This is why early Greek thought is rife with crackpot claims such as that motion is impossible (74) or “that man is all air, or fire, or water, or earth” (92). Indeed, the format demands a multiplicity of such viewpoints in competition with one another, whence “the remarkable proliferation of theories dealing with the same central issues” that “may well be considered one of the great strengths ... of Presocratic natural philosophy” (97).

The stage debate also explains why these theses were invariably defended by abstract deductive reasoning. “Given an interested but inexperienced audience, technical detail, and even careful marshalling of data, might well be quite inappropriate, and would, in any event, be likely to be less telling than the well-chosen plausible—or would-be demonstrative—argument.” (98) Hence we understand why “with the Eleatics logos—reasoned argument—comes to be recognised explicitly as *the* method of philosophical inquiry.” (78) This “notion of the supremacy of pure reason may ... be said to have promoted some of the triumphs of Greek science.” (121)

However, these triumphs of reason “were sometimes bought at the price of ... a certain impoverishment of the empirical content of the inquiry” (121). In early Greek science, “observations are cited to illustrate and support particular doctrines, almost, we might say, as one of the dialectical devices available to the advocates of the thesis in question.” (221) Also, “observations and tests ... could be deployed ... destructively ... [to disprove an opponent’s thesis], as they were by Aristotle especially, with great effect.” (224) These uses of observation fit well within the stage debate format. However, “theories were not put at risk by being checked against further observations carried out open-endedly and without prejudice as regards the outcome.” (222) We can understand why since “The speaker’s role was to advocate his own cause, to present his own thesis in as favourable a light as possible. It was not his responsibility to scrutinise ... the weaknesses of his own case with the same keenness with which he probed those of his opponent.” (98)

Of course, everyone was well aware of the deceptive potential of sly rhetoric for “making the worse argument appear the better” (99)—so much so that “early on it became a commonplace to insist on your own lack of skill in speaking” (100). But the Greeks did not see this as a reason to abandon the stage disputation format—instead they focussed on explicating “the correct rules of procedure for conducting a dialectical inquiry” (101).

Why did these things arise in Greece and not elsewhere? “Aristotle associated the development of speculative thought with the leisure produced by wealth,” (236) and indeed “the level of technology and that of economic development [in ancient Greece] were far in advance of those of many modern non-industrialised societies” (266). However, “Egypt and Babylonia ... were, economically, incomparably more powerful than any of the Greek city-states” (236), so the explanation for the “additional distinctively Greek factor” of “generalised scepticism” and “critical inquiry directed at fundamental issues” (264) must be sought elsewhere.

The answer may lie in “a particular social and political situation in ancient Greece, especially the experience of radical political debate and confrontation in small-scale, face-to-face societies. The institutions of the city-state ... put a premium on skill in speaking and produced a public who appreciated and the exercise of that skill. Claims to particular wisdom and knowledge in other fields besides the political were similarly liable to scrutiny, and in the competition between many and varied new claimants to such knowledge those who deployed evidence and argument were at an advantage compared with those who did not.” (266)

G. E. R. LLOYD, *Adversaries and Authorities: Investigations into Ancient Greek and Chinese Science*, Cambridge University Press, 1996.

“Any acquaintance with early Greek natural philosophy immediately brings to light a very large number of instances of

philosophers criticising other thinkers.” (21) Being a philosopher means being “subjected to blistering attack” (22). “From the list of occasions when philosophers are attacked by name ..., one could pretty well reconstruct the main lines of the development of Hellenistic philosophy itself.” (22-23) “Nor is this just a matter of ‘philosophy’, howsoever understood.” (23) There is “hard-hitting polemic” (23) in mathematics, medicine, and art as well. There is a “lack of great authority figures”; even Homer “is attacked more often than revered” (24).

The Greek style of philosophy is connected to its social context. “Greek pupils could and did pick and choose between teachers. ... Direct criticism of teachers is possible, and even quite common. ... Argument and debate ... are one of the means of attracting and holding students, and secondly they serve to mark the ... boundaries of groups.” (35) “The Greek schools were there not just, and not even primarily, to hand on a body of learned texts, but to attract pupils and to win arguments with their rivals. They may even be said to have needed their rivals, the better to define their own positions by contrast with theirs.” (38) “Dialectical debate, on which the reputations of philosophers and scientists alike so often depended, stimulated, when it did not dictate, confrontation. ... The recurrent confrontations between rival masters of truth left little room for the development of a consensus, let alone an orthodoxy, little sense of the need or desirability of a common intellectual programme.” (137)

“It was the rivalry between competing claimants to intellectual leadership and prestige in Greece, that stimulated the analysis of proving and of proof.” (57) “Many have assume that the internal dynamic of the development of mathematics itself would, somehow inevitably, eventually lead to a demand for strict axiomatic-deductive demonstration, and that there is accordingly no need to postulate any external stimulus such as I have conjectured ... Yet the difficulty for that view is ... [that] other, non-Greek, ancient mathematical traditions — Babylonian, Egyptian, Hindu, Chinese — ... all got along perfectly well without any notion corresponding to axioms and ... the particular notion of strict demonstration that went with it.” (58)

The underlying cause is perhaps captured by an admittedly “simplistic” dichotomy between “adversarial Greeks and irenic, authority-bound, Chinese” (44).

The different philosophical styles of ancient Greece and China reflect differences in their political systems. “Extensive political and legal debates, in the assemblies, councils and law-courts, were ... a prominent feature of the life of Greek citizens.” (74) “Greek philosophical and medical schools used, as the chief means for the expression of their own ideas and theories, both lectures ... and open, often public debates, sometimes modelled directly on the adversarial exchanges so familiar in Greek law-courts and political assemblies.” (39)

“So far from positively delighting in litigation, as many Greeks seem to have done, so far from developing a taste for confrontational argument in that context and becoming quite expert in its evaluations, the Chinese avoided any brush with the law as far as they could. Disputes that could not be resolved

by arbitration were felt to be a breakdown of due order and as such reflect unfavourably on both parties, whoever was in the right.” (220)

“The ... typical target audience envisaged in Greek rhetoric is some group of fellow citizens” (79), just as “In Greek law-courts the decisions rested with the ‘dicasts’ ... [who] were chosen by lot ... [and] combined the roles of both judge and jury.” (79) “In China, the [intended] audience for much philosophical and scientific work was very different: the ruler or emperor himself.” (39) “The Chinese were never in any doubt that the wise and benevolent rule of a monarch is the ideal.” (43)

“We often find Greek philosophers adopting a stance of fierce independence vis-a-vis rulers. ... With this independence came ... a disadvantage. Compared with their Chinese counterparts, Greek philosophers and scientists had appreciably less chance of having their ideas put into practice. ... Autocrats ... — as ... in China — ... could and did move swiftly from theoretical approval to practical implementation.” (43) “The superiority of theory to practice is a theme repeatedly taken up by scientists as well as philosophers in Greece: but that was sometimes to make a virtue out of necessity.” (44)

“Unlike in classical Greece, ... the bid to consolidate a comprehensive unified world-view was largely successful in China.” (116) “The prime duty of members of a Chinese Jia was the preservation and transmission of a received body of texts. In that context, pupils did not criticise teachers, and any given Jia did not see it as a primary task to take on and defeat other Jia in argument.” (44-45) While the Greeks “adopted a stance of aggressive egotism in debate, the tactics of Chinese advisers was rather to build on what could be taken as common ground, [and] certainly on what could be represented as sanctioned by tradition.” (221-222) “The emphasis is not on points at which [earlier philosophers] disagreed, ... but rather on what each of them had positively to contribute, how each succeeded, at least in part, in grasping some part of the Dao.” (25)

ÁRPÁD SZABÓ, *The Beginnings of Greek Mathematics*, Springer, 1978.

“Mathematics itself grew out of the more ancient subject of dialectic” (245). “Quite obviously it was dialectic which came first. The mere fact that all those terms which relate to the foundations of mathematics are of dialectical origin should have led us to this conclusion. Dialectic did not borrow any of its vocabulary from mathematics; instead, perfectly ordinary expressions from dialectic were transformed into technical mathematical terms. Hence early Greek mathematics, at least when it is viewed as an elaborately constructed system of knowledge, can properly be called a branch of dialectic.” (253-254)

The terms referred to here are those for axiom, postulate, hypothesis, etc., all of which, in dialectic, essentially stood for some variant of “concessions which the participants in a discussion have agreed to make” (238). Even in the best sources

the uses of such terms are not very systematic. For example, a clear distinction between definition and axiom is often lacking. On this basis “we can ... conjecture that the earliest foundations were composed only of definitions. After all, we know that the word *hypotheseis* (as a mathematical term) had two meanings; it could denote either a fundamental principle or just a definition. This in itself seems to indicate that fundamental principles were at one time identified with definitions. Furthermore, it is obvious that the very first *hypotheseis* on which the partners in a dialectical debate have to agree take the form of definitions.” (255)

“We know that the term *aitema* [=postulate] came from dialectic where it was used to denote a ‘demand’ about which the second partner in a dialogue had reservations. Let us see whether there is any connection between this early meaning of the word and Euclid’s postulates. At first glance, Postulates 1-3 appear to be such simple, self-evident and easily fulfilled ‘demands’ that one is tempted to disregard the literal meaning of their name.” (276) But they involve movement (of compasses etc.), which is a problematic notion especially in the view of Zeno and the Eleatics. “If we bear this in mind, it is easy to understand why Euclid’s first three postulates had to be laid down. ... They really are demands (*aitemata*) and not agreements (*homologema*); for they postulate motion, and anyone who adhered consistently to Eleatic teaching would not have been able to accept statements of this kind as a basis for further discussion.” (279)

“Our text of Euclid” has a separate heading called common notions, but this was not a well-entrenched term and these principles “obviously bore the name *axioma* in pre-Euclidean times” (281), and “the noun *axioma*, when used as a dialectical term, was originally synonymous with *aitema* and just meant a ‘demand’ or ‘request’” (286). Indeed, Euclid’s common notions “all assert properties of a relation (equality) which must have been regarded (by the Eleatics at least) as ‘self-contradictory’. It is by no means evident that two distinct things (i.e. two things which are not the same) can ever be ‘equal to one another’; furthermore, it makes no sense to speak of two things unless they can be distinguished from one another in some way.” (290)

The common notions are also dubious in that they “are assertions which are justified by practical experience and, in some cases, directly by sense-perception. Axiom 7, for example, states that ‘things which coincide with one another are equal to one another’. It can literally be seen that plane figures which coincide are actually equal; hence this axiom is verified by sensory experience.” (290) Therefore the common notions “could not have been accepted by the Eleatics, who required that all knowledge be obtained by purely intellectual means and without appealing to the senses. These principles were originally called demands (*axiomata*) because the other party in a dialectical debate had reservations about accepting them as a basis for further inquiry or, in other words, because their acceptance could only be demanded.” (301)

“After Plato’s time, however, the essentials of Eleatic dialectic were no longer very well understood; hence the ancient term *axioma* acquired a new meaning. Since it had always been

used to refer to a group of principles which, from the viewpoint of common sense, were evidently valid, it came now to denote those statements whose truth was ‘accepted as a matter of course’.” (301)

Early geometry may well have been based on visual reasoning. For example, the original meaning of the word for “prove” (*deiknymi*) was “show” or “point out” in a visual sense (although “we also find *deiknymi* used with the meaning ‘to make known by words’ in texts as early as the *Odyssey*” (188)). “Mathematics was at one time a practical science whose subject matter was empirical and whose principles were established by empirical methods”; indeed “the sudden turn away from empiricism [in mature Greek mathematics] seems somewhat surprising” (216): one must “explain why a predominantly empirical mathematical tradition suddenly and for no apparent reason became anti-empirical and anti-visual” (217).

This radical break was perhaps due to “the decisive influence of the Eleatic school of philosophy” (217). The notion of proof by contradiction provides the link between mathematics and the Eleatics. “It is apparent that indirect arguments played a very important part in Eleatic philosophy. Without them it would not have been possible to establish such central doctrines as that there is no motion, no change, no becoming, no perishing, no space and no time. Of course, these doctrines contradict the evidence of our senses and are incompatible with empiricism, nonetheless the Eleatics, bolstered by their belief that reason was the only guide to truth, accepted them.” (218)

Mathematicians were drawn into this school of philosophy since “the existence of incommensurability could not be conclusively proved by practical or empirical methods. Hence a complex of problems associated with incommensurability made it necessary to adopt Eleatic techniques of proof in geometry.” (316-317) Thus the earlier sense of proof, with its visual connotations, was replaced by axiomatic foundations in the sense of dialectic, and anti-empiricism came with the bargain.

KEN ALBALA, *Food: A Cultural Culinary History*, The Great Courses Plus, 2017.

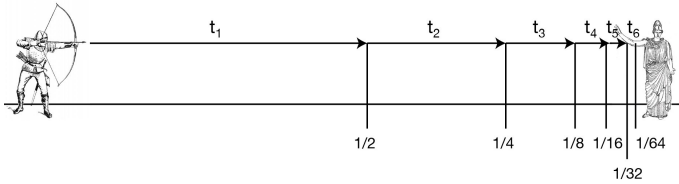
Islands $\Rightarrow$ good conditions for sea trade; independent city states. Hilly landscapes $\Rightarrow$ olives, grapes $\Rightarrow$ oil, wine = expensive non-perishables $\Rightarrow$ trade wealth $\Rightarrow$ large middle class with leisure time.

H. D. LEE (ED.), *Zeno of Elea: A Text, with Translation and Notes*, Cambridge University Press, 1936.

Zeno (c. -450) argued that motion is impossible. His “dichotomy” argument says:

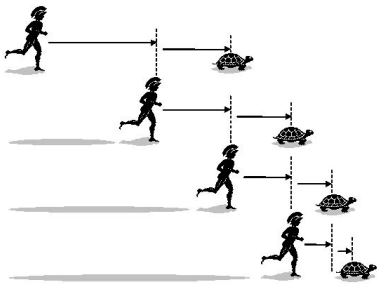
“An object in motion must move through a certain distance; but since every distance is infinitely divisible the moving object must first traverse half the distance through which it is moving,

and then the whole distance; but before it traverses the whole of the half distance, it must traverse half of the half, and again the half of this half. If then these halves are infinite in number, because it is always possible to halve any given length, and if it is impossible to traverse an infinite number of positions in a finite time ... [then] therefore it is impossible to traverse any magnitude in a finite time.” (45; Simplicius 1013.4)



There is also the “Achilles” form of the argument:

“The argument is called the Achilles because of the introduction into it of Achilles, who, the argument says, cannot possibly overtake the tortoise he is pursuing. For the overtaker must, before he overtakes the pursued, first come to the point from which the pursued started. But during the time taken by the pursuer to reach this point, the pursued always advances a certain distance; even if this distance is less than that covered by the pursuer, because the pursued is the slower of the two, yet none the less it does advance, for it is not at rest. And again during the time which the pursuer takes to cover this distance which the pursued has advanced, the pursued again covers a certain distance ... And so, during every period of time in which the pursuer is covering the distance which the pursued ... has already advanced, the pursued advances a yet further distance; for even though this distance decreases at each step, yet, since the pursued is also definitely in motion, it does advance some positive distance. And so ... we arrive at the conclusion that not only will Hector never be overcome by Achilles, but not even the tortoise.” (51; Simplicius 1014.9)



§ 6. Elements Defs.–Prop. 3: foundations; ruler and compass

6.1. Each of Euclid’s definitions may be considered a characterisation of a certain class of entities X. But in what sense? Match each of the following types of definitions with the best example from Euclid.

- ☐ “Test-condition” definition that enable us to answer, for any given object(s), the question “is this an instance of X?” in an unambiguous way that guarantees us to reach a yes or no answer by following a

practically executable set of steps.

- ☐ “Exclusion” definition that specifies X negatively, in terms of what it is not.
- ☐ “Fiat property” definition that characterise X in terms of its properties without being straightforwardly testable. In other words, it allows us to infer that if something is an X then it has such-and-such properties.
- ☐ “Maker’s” definition that implies a way of producing an X.
- ☐ “Shorthand” definition that give a convenient label to an entire set of properties or conditions.
- ☐ “Psychological” definition that give you a hunch what it is about without being mathematically exact.
- ☐ Definitions 1, 4, 10, 15, 22, 23

6.2. Match the concepts with how it would most naturally be defined.

- ☐ Test-condition, Fiat property, Maker’s, Shorthand, Psychological, Exclusion
- ☐ prime number, magnetic, cousin, stack [of pancakes, for example], intuitive, unicorn

6.3. Euclid’s definition of a straight line is vague. Can we interpret it to say: a straight line is a line such that any piece of it fits anywhere else on the line?

- ☐ Yes
- ☐ No

6.4. Euclid’s Proposition 1 has a logical gap or hidden assumption in it, namely that:

- ☐ C exists
- ☐ AC, BC are straight
- ☐ the two circles have the same radius
- ☐ the triangle produced is unique when in fact there are two

6.5. The gap in Euclid’s Proposition 1 could be resolved or addressed at least in part by an argument based on:

- ☐ Carrying out the constructions with ruler and compass.
- ☐ Allowing conclusions based on what is visually clear in the diagram, as long as it concerns not exact properties but only properties that would still hold even if the diagram was imperfectly drawn.
- ☐ The notions of “inside” and boundary in Definitions 13-15.

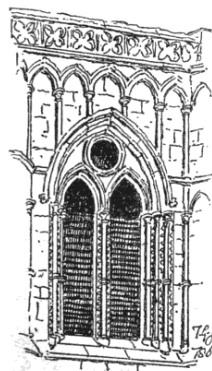
6.6. Propositions 2 and 3 suggest that:

- ☐ Postulate 3 is more restricted than one might think.

- Euclid's compass is "collapsible": when lifted from the paper, it collapses and forgets the radius it was set to.
- Euclid's compass is "rusty": once set, it's stuck at a particular opening.
- Euclid's ruler is unmarked, just a "straightedge," that can be used to draw lines but not to measure lengths.
- Euclid's ruler can be used to produce equal line segments (as if you can put a single mark on it corresponding to the segment you have), but not to produce segments of other sizes (as if you had a full numerical scale on it).

6.7. In the Declaration of Independence of the United States, the first part of the second sentence has a very Euclidean ring to it. It is reminiscent of one of Euclid's postulates in particular—which one? Postulate □. There are in fact several further allusions to Euclidean rhetoric in this document.

6.8. This is a typical window design in Gothic architecture:



This design is reminiscent of Euclid's Proposition □. The Gothic style of architecture arose in the early 12th century, within a decade or two of the first Latin translation of Euclid's *Elements*.

PROCLUS, *A Commentary on the First Book of Euclid's Elements*, c. 450, translated by Glenn R. Morrow, Princeton University Press, 1992.

"Proclus' commentary on book I of Euclid's *Elements* is almost certainly a written version of lectures which he presented to students and associates in Athens in the mid-fifth century. ... Readers of the commentary should always bear in mind that, although it is the work of Proclus, it is also a record of an educational and intellectual tradition." (ix)

Mathematics stems from the soul, not sense experience. "Should we admit that [the objects of mathematics] are derived from sense objects, either by abstraction, as is commonly said, or by collection from particulars to one common definition?" (10) No, because "The unchangeable, stable, and incontrovertible character of the propositions [of mathematics] shows that it is superior to the kinds of things that move about in matter." (3) "And how can we get the exactness of our precise and irrefutable concepts from things that are not precise? ... We must therefore posit the soul as the generatrix of mathematical forms and ideas." (11)

Nevertheless mathematics has many uses. For although we just argued that it is "immaterial and theoretical; when it touches on the material world it delivers out of itself a variety of sciences—such as geodesy, mechanics, and optics—by which it benefits the life of mortals. ... and many things incredible to men it has made credible to all. Recall what Hieron of Syracuse is said to have remarked about Archimedes, who had built a three masted vessel ... When all the Syracusans together were unable to launch it and Archimedes made it possible for Hieron alone to move it down to the shore [by a system of pulleys], he exclaimed, in his amazement: 'From this day forth we must believe everything that Archimedes says.' ... Many of our predecessors have recorded such things in praise of mathematics." (50-51)

"There are nevertheless contentious persons who endeavor to detract from the worth of this science, ... declaring that the empirical sciences concerned with sense objects are more useful than the general theorems of mathematics. Mensuration, they say, is more useful than geometry, popular arithmetic than the theory of numbers, and navigation than general astronomy. For we do not become rich by knowing what wealth is but by using it, nor happy by knowing what happiness is but by living happily. Hence we shall agree, they say, that the empirical sciences, not the theories of the mathematicians, contribute most to human life and conduct. Those who are ignorant of principles but practiced in dealing with particular problems are far and away superior in meeting human needs to those who have spent their time in the schools pursuing theory alone." (22)

Reply to this objection. "We do not think it proper ... to measure its utility by looking to human needs and making necessity our chief concern. ... We must therefore posit mathematical knowledge and the vision that results from it as being worthy of choice for their own sakes, and not because they satisfy human

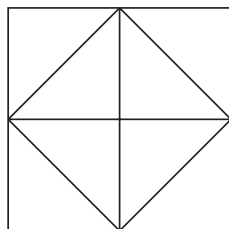
§ 7. Plato and Aristotle on geometry

- 7.1. ☞ What makes mathematical knowledge special? What sets it apart from other fields?
- 7.2. ☞ In the Platonic worldview (shared by Proclus), what is the relation between mathematics and physical reality? What are strengths and weaknesses of this view?
- 7.3. ☞ What aspects of technical mathematics do the authors below seem to have had especially in mind?
- 7.4. ☞ In Rafael's famous fresco "The School of Athens," Plato is pointing toward the sky and Aristotle is pointing straight ahead. Why?
- 7.5. ☞ Are there examples of theories in which the principles are not primitives, or the primitives are not principles, in Aristotle's sense? (Cf. §20.)

needs. And if we must relate their usefulness to something outside them, it is to intellectual insight that they must be said to be contributory. For to that they lead the way and prepare us by purifying the eye of the soul and removing the hindrances that the senses present to our knowing the whole of things. ... Consequently instead of crying down mathematics for the reason that it contributes nothing to human needs ... we should, on the contrary, esteem it highly because it is above material needs and has its good in itself alone.” (23-24)

PLATO, *Meno*, translated by W. K. C. Guthrie, Penguin, 1956.

Socrates wants to illustrate the nature of mathematical knowledge by leading an uneducated slave boy to discover how to double a given square of area four square feet. In the course of the dialog below he draws this figure in the sand, starting with one of the four small squares:



“SOCRATES: Tell me, boy, is not this our square of four feet? You understand? BOY: Yes. SOCRATES: Now we can add another equal to it like this? BOY: Yes. SOCRATES: And a third here, equal to each of the others? BOY: Yes. SOCRATES: And then we can fill in this one in the corner? BOY: Yes. SOCRATES: Then here we have four equal squares? BOY: Yes. SOCRATES: And how many times the size of the first square is the whole? BOY: Four times. SOCRATES: And we want one double the size. You remember? BOY: Yes. SOCRATES: Now does this line going from corner to corner cut each of these squares in half? BOY: Yes. SOCRATES: And these are four equal lines enclosing this area? BOY: They are. SOCRATES: Now think. How big is this area? BOY: I don't understand. SOCRATES: Here are four squares. Has not each line cut off the inner half of each of them? BOY: Yes. SOCRATES: And how many such halves are there in this figure? BOY: Four. SOCRATES: And how many in this one? BOY: Two. SOCRATES: And what is the relation of four to two? BOY: Double. SOCRATES: How big is this figure then? BOY: Eight feet. SOCRATES: On what base? BOY: This one. SOCRATES: The line which goes from corner to corner of the square of four feet? BOY: Yes. SOCRATES: The technical name for it is ‘diagonal’; so if we use that name, it is your personal opinion that the square on the diagonal of the original square is double its area. BOY: That is so, Socrates. SOCRATES: What do you think, Meno? Has he answered with any opinions that were not his own? MENO: No, they were all his. SOCRATES: Yet he did not know, as we agreed a few minutes ago. MENO: True. SOCRATES: But these opinions were somewhere in him, were they not? MENO: Yes. SOCRATES: So a man who does not know has in himself true opinions on a subject without having

knowledge. ... This knowledge will not come from teaching but from questioning. He will recover it for himself.” (84d-85d)

PLATO, *Republic*, c. –380, quoted from *Complete Works*, ed. John M. Cooper, Hackett, 1997.

“We must require those in your fine city not to neglect geometry in any way, for even its byproducts are not insignificant. ... When it comes to better understanding any subject, there is a world of difference between someone who has grasped geometry and someone who hasn’t.”

But: “No one with even a little experience of geometry will dispute that this science is entirely the opposite of what is said about it in the accounts of its practitioners. ... They give ridiculous accounts of it, ... for they speak like practical men, and all their accounts refer to doing things. They talk of ‘squaring,’ ‘applying,’ ‘adding,’ and the like, whereas the entire subject is pursued for the sake of knowledge ... [and] for the sake of knowing what always is, not what comes into being and passes away.” (VII, 527)

PLATO, *Timaeus*, translated by Donald J. Zeyl, Hackett, 2000.

Human anatomy is an appendix to the soul. “The entire body” was created “as its vehicle” (69c), and its properties were designed to serve the soul, e.g., “They wound the intestines round in coils to prevent the nourishment from passing through so quickly that the body would of necessity require fresh nourishment just as quickly, there by rendering it insatiable. Such gluttony would make our whole race incapable of philosophy and the arts, and incapable of heeding the most divine part within us.” (73a). Even eyesight was created not for worldly purposes but primarily to give us the mind the idea of number and time:

Origins of human understanding. “Our ability to see the periods of day-and-night, of months and of years, of equinoxes and solstices, has led to the invention of number and has given us the idea of time and opened the path to inquiry into the nature of the universe. These pursuits have given us philosophy, a gift from the gods to the mortal race whose value neither has been nor ever will be surpassed. I’m quite prepared to declare this to be the supreme good our eyesight offers us.” (47a-b).

Generation of animals. “[Birds] descended from ... simple-minded men, men who studied the heavenly bodies but in their naiveté believed that the most reliable proofs concerning them could be based upon visual observation. Land animals ... came from men who had no tincture of philosophy and who made no study of the heavens whatsoever ... As a consequence ... they carried their forelimbs and their heads dragging toward the ground.” (91d-92a).

The essence Aristotle's view of the axiomatic-deductive method is summed up in the following sentence: "Demonstrative understanding ... must proceed from items which are true and primitive and immediate and more familiar than and prior to and explanatory of the conclusions." (71b)

Three notable consequences of this thesis are:

The axiomatic-deductive method is much more than mere logic. "There can be a deduction even if these conditions are not met, but there cannot be a demonstration—for it will not bring about understanding." (71b)

There is a fundamental distinction between "demonstrations which are said to demonstrate and those which lead to the impossible" (85a), i.e., proofs by contradiction, which must be seen as intrinsically inferior (87a).

Axioms stem from perception. "I call prior and more familiar in relation to us items which are nearer perception" (72a), so immediate perception must be the ultimate foundations of "demonstrative understanding." "We must get to know the primitives [i.e., axioms] by induction; for this is the way in which perception instills universals." (100b) However, "for the principles [i.e., axioms] a geometer as geometer should not supply arguments" (77b). Note the two coextensive words for "axiom"—indeed, "I call the same things principles and primitives" (72a), since logical starting points of a deductive system and immediately given truths should be the same thing.

§ 8. Elements 4–11: triangle congruence; constructions

8.1. Propositions 1-3 were "problems" (showing how to do or make something) while Proposition 4 is a "theorem" (showing that a property or relation holds for certain objects). The difference is signalled by fixed stock phrases, most notably in the paragraph.

- ☐ First
- ☐ Second
- ☐ Third
- ☐ Last

8.2. Proposition 7. The proof doesn't work if:

- ☐ $AC=CB$ and $AD=DB$.
- ☐ C is on AD .
- ☐ C is inside triangle ADB .

8.3. Proposition 8. How can we conclude that the remaining pairs of angles are equal?

- ☐ It follows from the proof of Proposition 8 itself.

- ☐ It follows by applying Proposition 8 anew to the same triangles but with a different choice of "base".
- ☐ It can be obtained by applying Proposition 4.
- ☐ We can't; they may not be equal.

8.4. Why does Euclid bisect an angle (Proposition 9) before a length (Proposition 10)?

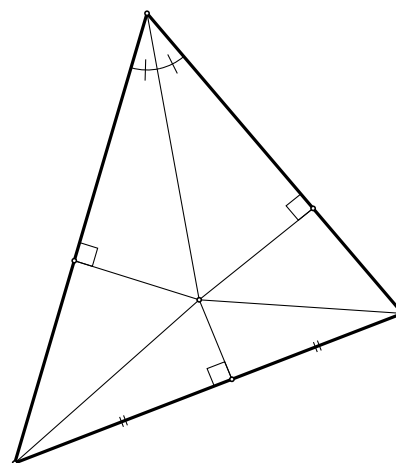
- ☐ Because the ruler-and-compass steps of the corresponding constructions proceed in this order.
- ☐ Because the only previous proposition that has equalities of sides in its conclusion is SAS.
- ☐ Because we have a postulate about equal angles (Postulate 4) but none about equal lengths.
- ☐ The order is an arbitrary choice since the two propositions are independent.

8.5. Proposition 9. Suppose I follow Euclid's construction up to the point when DE has been drawn. If I divided DE into three equal parts (assuming for the moment that I can do this somehow), and then connected the dividing points up to A , will I have cut the angle into three equal parts?

- ☐ Yes
- ☐ No

8.6. Consider the figure below. The point in the middle is defined as the intersection of the bisector of the top angle and the perpendicular bisector of the base. By construction, the top two triangles are congruent, and the two base triangles are congruent. It follows that the remaining two triangles are congruent. Which triangle congruence principles were needed to establish these three congruences?

- ☐ AAS
- ☐ ASA
- ☐ SAS
- ☐ SSS
- ☐ SSRA

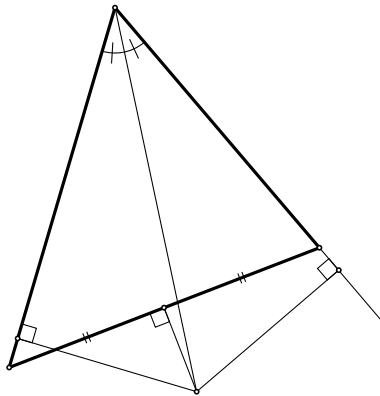
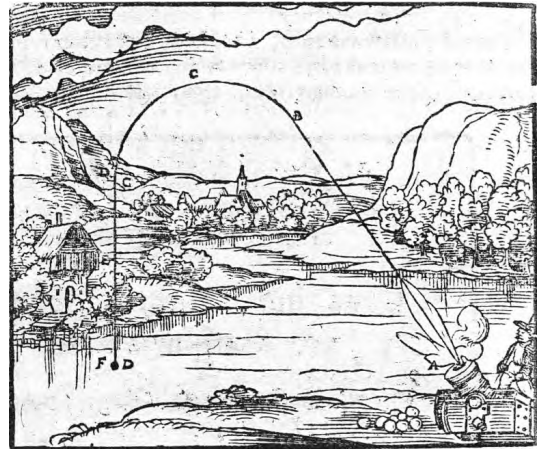


8.7. (cont.) Since we started with an arbitrary triangle, it follows that all triangles are:

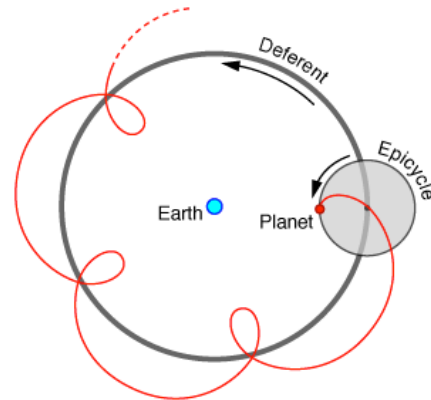
- ☐ right-angled
- ☐ isosceles
- ☐ bisected in area by perpendicular bisector of each side
- ☐ decomposable into six triangles with equal area

8.8. (cont.) Actually the figure was incorrectly drawn. Below is the correct figure. On the basis of this example one could argue that:

- ☐ It can be dangerous to rely on diagrams when writing geometrical proofs.
- ☐ Exact constructions are important to safeguard rigour in geometrical reasoning.



Similarly, planetary trajectories were for thousands of years always thought of as combinations of circular motions:



In Aristotelian physics, rectilinear or circular motion was taken to be the fundamental tendency of all matter: earth and water want to go straight down, fire and air want to go straight up, and heavenly bodies are made of a fifth element that wants to go in circles.

In mathematics, there has always been one serious competitor to ruler and compass who likewise makes a claim to being the true bedrock of mathematics. A rival to the throne, as it were. This rival is number. “All is number,” said the Pythagoreans, and with good reason. Still today this paradigm is alive and well. Indeed, it has arguably won over geometry: the modern way of thinking of geometry in terms of coordinate systems and equations is effectively tantamount to taking numbers to be more fundamental than shapes and motions.

But the mathematician’s worst nightmare was realised when they demonstrated to their own horror that $\sqrt{2}$ is in a sense not a number at all, namely:

- 9.1. Prove that $\sqrt{2}$ is irrational, that is to say, it is not the ratio of two integers. Hint: assume, on the contrary, that $\sqrt{2} = p/q$, where p and q are integers, and derive a contradiction.

Of course we can still say that “ $\sqrt{2}$ is the number such that when you multiply it by itself it becomes 2.” But then again we can also say that “a unicorn is a horse with big horn on its forehead.” Is $\sqrt{2}$ just as mythical as a unicorn? A fiction of the

§ 9. Ruler and compass

Euclid’s geometry is essentially the geometry of ruler and compasses. There are many reasons why these tools form a beautiful foundation for geometry. They embody the purest concepts of geometry: line and circle, straightness and length. They are extremely simple to produce instrumentally: a child can generate both with a piece of string.

It is perhaps a testament to our tendency to think in terms of these simplest categories that we have often imposed them on phenomena that are not actually of this nature at all. For example, many Renaissance texts conceptualise canon ball trajectories as made up of rectilinear and circular pieces:

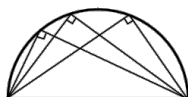
imagination? Surely mathematics must aspire to know its objects of investigation in a more definitive way than story books and fairytales.

So the irrationality of $\sqrt{2}$ was a devastating blow to the Number camp and their “all is number” slogan. Legend has it that one renegade Pythagorean was drowned at sea for pushing this point among believers in the arithmetical faith.

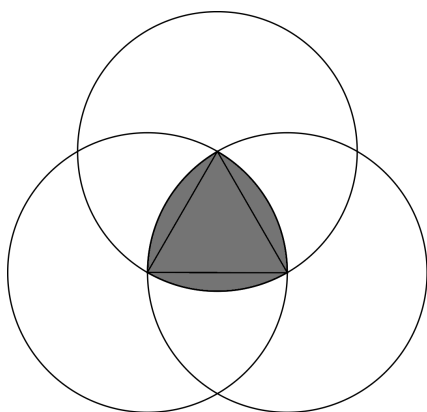
Meanwhile, ruler and compass triumph when faced with the same test. Adherents of this faction do not need to resort to unicorn-style definitions when asked what $\sqrt{2}$ is. They can simply draw a unit square, connect its diagonal, and say: *that's* $\sqrt{2}$.

More generally, we can construct any square root $\sqrt{a}$ as follows (cf. *Elements*, II.14). Draw a circle with diameter $a+1$ and draw the perpendicular at the dividing point between the a and 1 segments.

- 9.2. Explain how $\sqrt{a}$ can be obtained from the resulting figure. Hint: recall Thales' Theorem:



- 9.3. What is $\sqrt{7}$? Find out using only ruler and compass.
- 9.4. Consider the “fat triangle” (shaded in the picture below) obtained by adding one more circle to the construction of Euclid's Proposition 1.

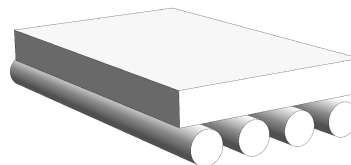


Which of the following are true? Hint: You may want to use cardboard cutouts to investigate this.

- ☐ If we measure the width of a “fat triangle” with a caliper, the result is the same in every direction.



- ☐ If the rollers in the setup below had “fat triangle” cross-sections, the platform would still roll as smoothly as before (without bobbing up and down).



- ☐ The “fat triangle” has a “midpoint” which is equidistant from the perimeter in every direction.

- 9.5. The website “Below the Surface” <https://belowthesurface.amsterdam/en/vondsten> catalogues items found at the bottom of an excavated canal in Amsterdam. Do a keyword search for “compasses.” Roughly from what year are the oldest compasses that were found in the canal? Perhaps they were thrown into the canal by frustrated geometry students.

- ☐ 1200
- ☐ 1400
- ☐ 1600
- ☐ 1800

§ 10. Constructivism

- 10.1. ☞ How did 17th-century philosophers interpret the purpose of constructions in Euclidean geometry?
- 10.2. ☞ What implications did they draw from this for philosophy and other branches of knowledge?

PAOLO SARPI, *Pensieri filosofici e scientifici*, no. 1, 1578.

“We know for certain both the existence and the cause of those things which we understand fully how to make ... [just as] in mathematics someone who composes [i.e. demonstrates synthetically] knows because he makes.”

GIAMBATTISTA VICO, *On the Study Methods of our Time*, translated from the 1709 original by Elio Gianturco. The Library of Liberal Arts, Bobbs-Merrill Company, 1965.

“We are able to demonstrate geometrical propositions because we create them; were it possible for us to supply demonstrations of propositions of physics, we would be capable of creating them *ex nihilo* as well.” (IV.23)

PIERRE GASSENDI, *Exercitationes*, 1624, quoted from *The Selected Works of Pierre Gassendi*, ed. and transl. Craig B. Brush, Johnson Reprint Co., New York, 1972.

“Things not yet created and having no existence, but being merely possible, have no reality and no truth.” (253) “The moment you pass beyond things that are apparent, or fall un-

der the province of the senses and experience, in order to inquire about deeper matters, both mathematics and all other branches of knowledge become completely shrouded in darkness." "[Hence mathematical objects must be] considered in actual things"; indeed, "as soon as numbers and figures are considered abstractly ... then they are nothing at all." (107)

JOHN AUBREY, *Brief Lives*, late 17th century.

"[Thomas Hobbes] was 40 years old before he looked on geometry; which happened accidentally. Being in a gentleman's library, Euclid's Elements lay open, and 'twas the 47 El. libri I [Pythagorean Theorem]. He read the proposition. By God, sayd he (he would now and then sweare an emphaticall Oath by way of emphasis), this is impossible! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. Et sic deinceps, that at last he was demonstratively convinced of that trueth. This made him in love with geometry."

THOMAS HOBBS, *Leviathan*, 1651. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume III.

"Geometry ... is the only science that it hath pleased God hitherto to bestow on mankind" (23-24).

"There can be nothing so absurd, but may be found in the books of philosophers. And the reason is manifest. For there is not one of them that begins his ratiocination from the definitions, or explications of the names they are to use; which is a method that hath been used only in geometry; whose conclusions have thereby been made indisputable." (33)

"For all men by nature reason alike, and well, when they have good principles. For who is so stupid, as both to mistake in geometry, and also to persist in it, when another detects his error to him?" (35)

"By Philosophy is understood the knowledge acquired by reasoning, from the manner of the generation of any thing, to the properties: or from the properties, to some possible way of generation of the same; to the end to be able to produce, as far as matter, and human force permit, such effects, as human life requireth. So the geometrician, from the construction of figures, findeth out many properties thereof; and from the properties, new ways of their construction, by reasoning; to the end to be able to measure land, and water; and for infinite other uses." (664)

THOMAS HOBBS, *De Corpore*, 1655. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume I.

"Philosophy is such knowledge of effects or appearances as we

acquire by true ratiocination from the knowledge we have first of their causes or generation." (3)

This definition is explicitly modelled on mathematics: "How the knowledge of any effect may be gotten from the knowledge of the generation thereof, may easily be understood by the example of a circle: for if there be set before us a plain figure, having, as near as may be, the figure of a circle, we cannot possibly perceive by sense whether it be a true circle or no ... [But if] it be known that the figure was made by the circumduction of a body whereof one end remained unmoved" then the properties of a circle become evident. (6)

Another way of putting it is that "The subject of Philosophy, or the matter it treats of, is every body of which we can conceive any generation." (10) Just as the domain of geometry is the set of constructible curves.

As the circle example shows, motion is a basic form of generation. Indeed motion is the foundation of geometry and physics alike. "First we are to observe what effect a body moved produceth, when we consider nothing in it besides its motion; ... from which kind of contemplation sprung that part of philosophy which is called geometry." (71) Next "we are to pass to the consideration of what effects one body moved worketh upon another; ... that is, when one body invades another body which is either at rest or in motion, what way, and with what swiftness, the invaded body shall move; and, again, what motion this second body will generate in a third, and so forwards. From which contemplation shall be drawn that part of philosophy which treats of motion ... [and ultimately] comprehend that part of philosophy which is called physics." (71-72)

"And, therefore, they that study natural philosophy, study in vain, except they begin at geometry; and such writers or disputers thereof, as are ignorant of geometry, do but make their readers and hearers lose their time." (73)

"After physics we must come to moral philosophy; in which we are to consider the motions of the mind, namely, appetite, aversion, love, benevolence, hope, fear, anger, emulation, envy, &c." (72) "For the causes of the motions of the mind are known ... And, therefore, ... by the synthetical method, and from the very first principles of philosophy, [one] may by proceeding in the same way [as in geometry and physics], come to the causes and necessity of constituting commonwealths, and to get the knowledge of what is natural right, and what are civil duties; and, in every kind of government, what are the rights of the commonwealth, and all other knowledge appertaining to civil philosophy." (73-74)

Carefully enunciated definitions are another prominent aspect of mathematics that is to be carried over into general philosophy. "Whatsoever the common use of words be, yet philosophers, who were to teach their knowledge to others, had always the liberty ... of taking to themselves such names as they please for the signifying of their meaning, if they would have it understood. Nor had mathematicians need to ask leave of any but themselves to name the figures they invented, parabolas, hyperboles, cissoeides, quadratrices, &c. or to call one magnitude A, another B." (16) "But definitions of things, which may be un-

derstood to have some cause, must consist of such names as express the cause or manner of their generation, as when we define a circle to be a figure made by the circumduction of a straight line in a plane, &c.” (81-82)

Logical, syllogistic reasoning is another distinctive attribute of mathematics. In fact, “They that study the demonstrations of mathematicians, will sooner learn true logic, than they that spend time part in reading the rules of syllogizing which logicians have made; no otherwise than little children learn to go, not by precepts, but by exercising their feet.” (54-55)

THOMAS HOBBS, *Six Lessons to the Savilian Professors of the Mathematics*, 1656. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume VII.

“Of arts, some are demonstrable, others indemonstrable; and demonstrable are those the construction of the subject whereof is in the power of the artist himself, who, in his demonstration, does no more but deduce the consequences of his own operation. The reason whereof is this, that the science of every subject is derived from a precognition of the causes, generation, and construction of the same; and consequently where the causes are known, there is place for demonstration, but not where the causes are to seek for. Geometry therefore is demonstrable, for the lines and figures from which we reason are drawn and described by ourselves; and civil philosophy is demonstrable, because we make the commonwealth ourselves.” (183-184)

- 10.3. ☞ Why did constructivist philosophies of the foundations of mathematics gain popularity in the 20th century?
- 10.4. ☞ In what ways can 20th-century constructivist philosophy be considered similar to what Euclid was doing?
- 10.5. ☞ Suppose I add to Euclid’s *Elements* the definition: “A superright triangle is a triangle each of whose angles is a right angle.” So its angle sum is three right angles. But we also know from Proposition 32 that the angle sum of a triangle is two right angles. Therefore $3 = 2$. How can we avoid this paradox? How can guarantee that mathematics is free of such contradictions?
- 10.6. ☞ How do we know that mathematics does not contain statements that can be proven both true and false?

WILLIAM KNEALE & MARTHA KNEALE, *The Development of Logic*, Oxford University Press, 1962, page 114.

[Eubulides in the fourth century BC] made ... important contributions to the development of logic, [including] the invention of a number of interesting paradoxes, [such as:] ...

The Liar. A man says that he is lying. Is what he says true or false? [If it’s true, then it must be false. If it’s false, then it must be true.] ...

The Horned Man. What you have not lost you still have. But you have not lost horns. So you still have horns.

L. E. J. BROUWER, *Over de Grondslagen der Wiskunde*, Ph.D. thesis, University of Amsterdam, 1907. Quoted from the translation in L. E. J. Brouwer, *Collected Works*, I, North-Holland, 1975.

Wheresoever in logic the word *all* or *every* is used, this word, in order to make sense, tacitly involves the restriction: *insofar as belonging to a mathematical structure which is supposed to be constructed beforehand*. (76)

By a propositional function ... the logicians ... mean a statement about x ...; they reckon that by that statement a class is defined, consisting of all things ... which by substitution make the statement true. [Thus they speak of] *all things for which the statement φx is true*. ... As the fundamental domain of operations, within which the relations meant by the words or symbols must exist, they choose not some mathematical system, but the chimerical ‘everything’. ... [They thus] give a linguistic system of statements and propositional functions priority over mathematics. (88) They postulate that these sentences define classes and that it is allowed to reason about these classes according to the laws of classical logic. (89)

It is not surprising that they ... came up against contradictions. ... Russell discusses most amply the following contradiction: ‘I consider the class of all classes which do not possess the property ... to occur amongst their own elements. Does this class possess the property in question? If it does, then it occurs amongst its own elements, so it is one of the classes which do not possess the property, so it does not have the property. And vice-versa: If it does not, then it is on equal terms with its own elements, so it does have the property.’ [Cf. Figure 3]

Russell suggests various methods to escape from the contradiction. ... ‘[One] suggestion’, he says, ‘would be to demur to the notion of *all* objects, but in any case the notion of *every object* must be retained, for there are truths, viz. the logical principles, which hold for every object.’

But this is mistaken: the logical principles hold exclusively for words with a mathematical content. And exactly because Russell’s logic is no more than a linguistic system, deprived of a presupposed mathematical system to which it would be related, there is no reason why no contradictions would appear. (89)

The language of Euclidean geometry ... is reliable only because the mathematical systems and relations, which are symbolized by the words of that language as conventional signs, have been constructed beforehand independently of that language. (98)

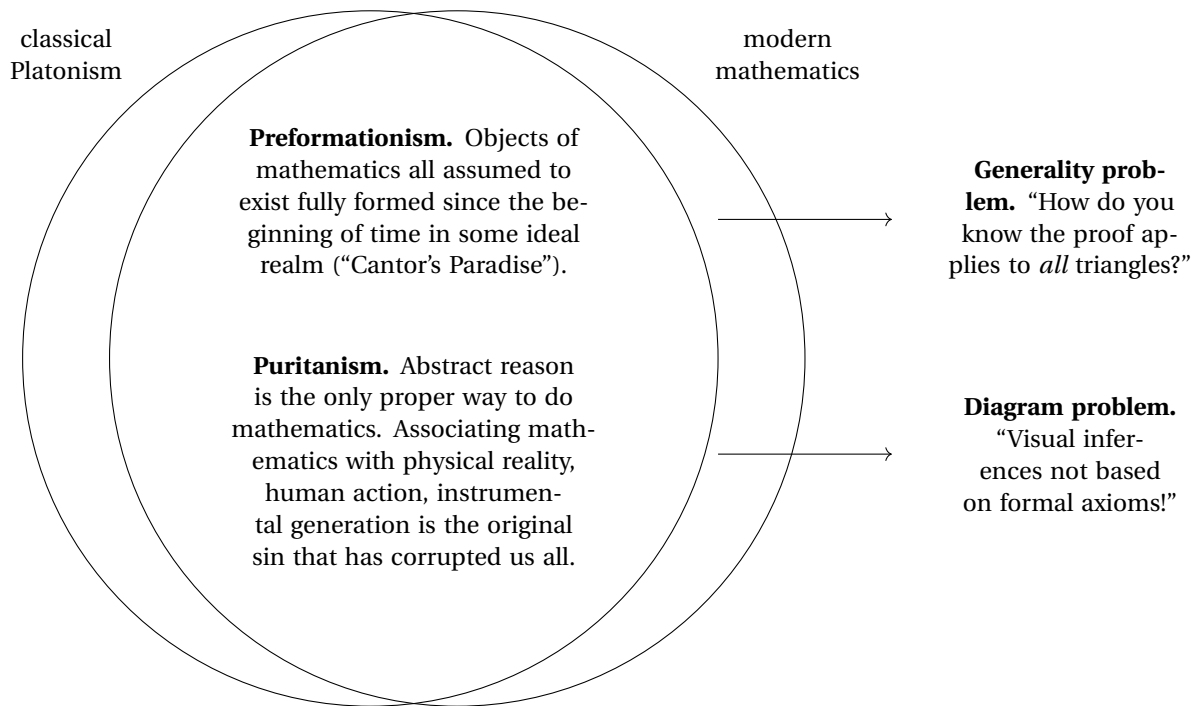


Figure 1: Shared commitments of ancient and modern Platonism.

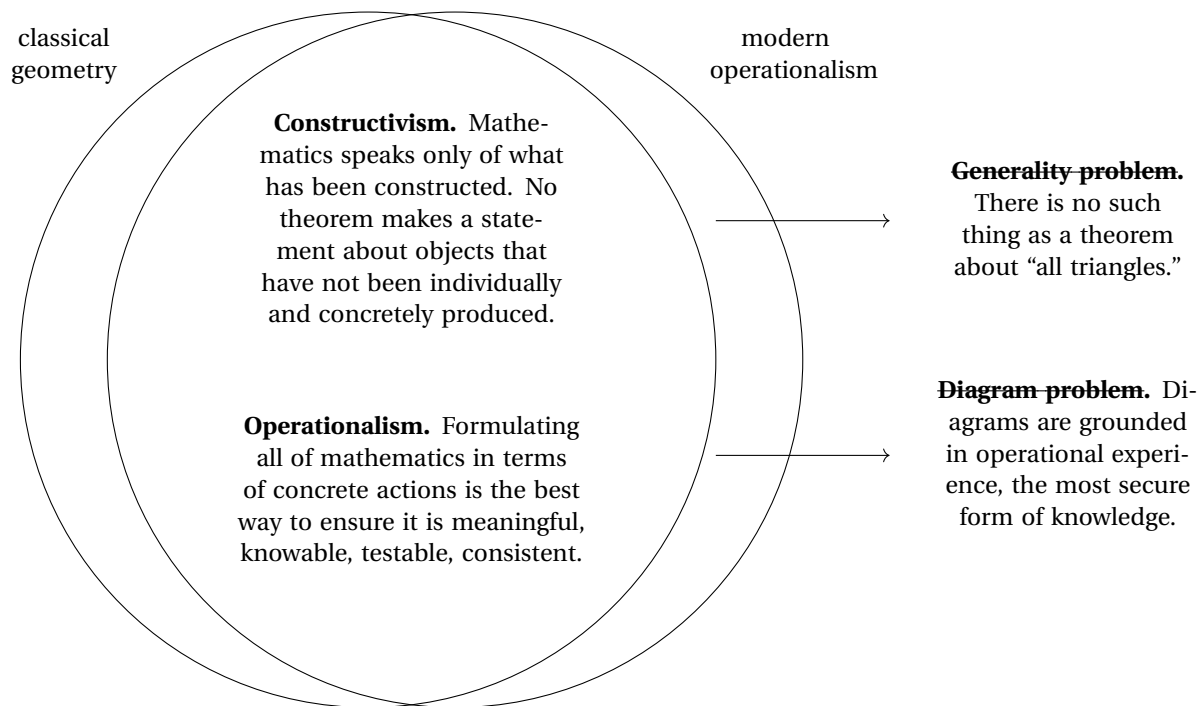


Figure 2: Shared commitments of ancient and modern operationalism.

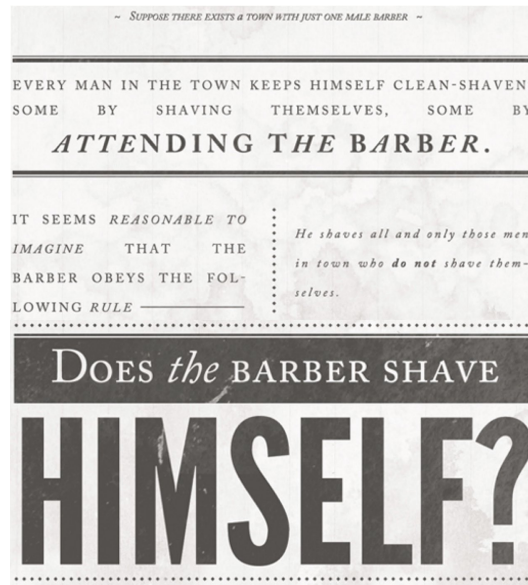


Figure 3: Popularised version of Russell's Paradox.

P. W. BRIDGMAN, *The Logic of Modern Physics*, New York: Macmillan, 1927. Paperback reprint 1960.

"We mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations." (5) "The proper definition of a concept is not in terms of its properties but in terms of actual operations." (6)

Motivation for this: Relativity theory and quantum physics show that "when experiment is pushed into new domains, we must be prepared for new facts, of an entirely different character from those of our former experience" (2). Therefore "the physicist ... recognizes no *a priori* principles which determine or limit the possibilities of new experience" (3); rather we "must use in describing and correlating nature concepts of such a character that our present experience does not exact hostages of the future" (4). "For if experience is always described in terms of experience, ... we need never be embarrassed" (6–7) to find that we were talking about imagined concepts with no connection to reality.

Newton's concept of absolute time violates these principles. (4–7)

"[A] consequence of the operational character of our concepts ... is that it is quite possible, even disquietingly easy, to invent expressions or to ask questions that are meaningless. It constitutes a great advance in our critical attitude toward nature to realize that a great many of the questions that we uncritically ask are without meaning. If a specific question has meaning, it must be possible to find operations by which an answer may be given to it." (28)

P. W. BRIDGMAN, *The nature of some of our physical concepts*, New York: Philosophical Library, 1952.

"We do not know the meaning of a concept unless we can specify the operations which were used by us or our neighbour in applying the concept in any concrete situation" (7)

"Physicists do profitably employ concepts the meaning of which is not to be found in the instrumental operations of the laboratory, and which cannot be reduced to such operations without residue. Nearly all the concepts of theoretical and mathematical physics are of this character. ... In fact, there is hardly any physical concept which does not enter to a certain extent into some theoretical edifice and which does not therefore possess to a certain degree a non-instrumental component." (8)

"I think, however, that physicists are agreed in imposing one restriction on the freedom of such operations, namely that such operations must be capable of eventually, although perhaps indirectly, making connection with instrumental reality. Only in this way can the physicist keep his feet on the ground or achieve a satisfactory degree of precision. ... Politics, philosophy and religion are full of ... purely verbal concepts; it is merely that such concepts are outside the field of the physicist." (10)

P. W. BRIDGMAN, *Reflections of a Physicist*, second ed., New York: Philosophical Library, 1955.

"In mathematics much the same situation arises that arises in physics; in foundation studies, in which one wants to secure the maximum awareness of what one is doing and the max-

imum security that he is not involving himself in contradiction, one would do well to use only concepts whose meaning is found in ... unambiguously performable operations. For in this way, ... the description of a situation in mathematics reduces to the description of an actual experience, ... and actual experience in not self-contradictory." (17)

"One of the chief advantages of making definitions operational is that it increases precision. I find that I myself use terms without either precisely formulating the conditions which demand their use. ... When my neighbour uses the term, I observe only too frequently that I cannot deduce from his use of it the occurrence of certain antecedent or subsequent occurrences which I believe are implied in his use of it. In such cases reduction of definitions to operational terms increases precision, so that operational definitions are called for when increased precision is called for." (30)

Operationalism implies questioning the Platonic philosophy of mathematics, e.g. that numbers "exist." (100)

"The very nature of meaning itself makes it impossible to get away from the human reference point" (112)

"If I am trying to describe a situation, the criterion that I have successfully communicated it, is that my fellow be able to *re-construct* the situation. ... The meaningfulness of the communication is therefore to be found in the actions which accompany it." (121)

§ 11. Mathematics and culture in early modern Europe

11.1. ☞ What is the mathematical justification for Alberti's recipe for drawing a tiled floor?

11.2. ☞ In what way has perspective painting played a role in broader philosophical and scientific developments?

VITRUVIUS, *De Architectura*, 1st century BC, VII.11, quoted from Vitruvius, *The Ten Books on Architecture*, translated by Morris Hicky Morgan, Harvard University Press, 1914, 198.

Geometrical principles of perspective can be used to deceive the eye. "By this deception a faithful representation of the appearance of buildings might be given in painted scenery, ... so that, though all is drawn on a vertical flat facade, some parts may seem to be withdrawing into the background, and others to be standing out in front."

PLATO, *Republic*, c. -380, quoted from *Complete Works*, ed. John M. Cooper, Hackett, 1997.

"*Trompe l'œil* painting" has "powers that are little short of magical," "because they exploit this weakness in our nature," by-

passing "the rational part of the soul." (X.602d) The solution to this problem, as Plato saw it, was a solid mathematical education. Since "sense perception seems to produce no sound result" with these "*trompe l'œil* paintings" (VII.523b), "it makes all the difference whether someone is a geometer or not" (VII.526d). "There's no knowledge of ... sensible things, whether by gaping upward or squinting downward." (VII.529b) For example, "let's study astronomy by means of problems, as we do geometry, and leave the things in the sky alone." (VII.530b)

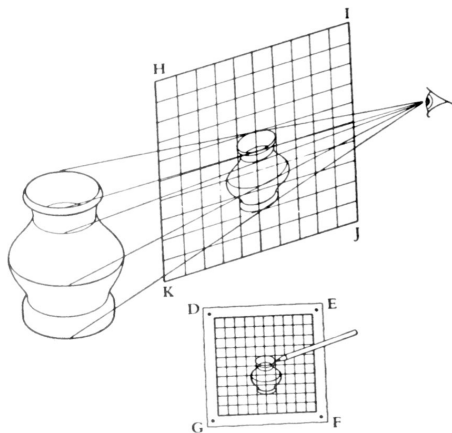
PLATO, *Protagoras*, c. -380, quoted from *Complete Works*, ed. John M. Cooper, Hackett, 1997.

"The power of appearance often makes us wander all over the place in confusion, often changing our minds about the same thing and regretting our actions and choices with respect to things large and small." "The art of measurement," by contrast, "would make the appearances lose their power" and "give us peace of mind firmly rooted in the truth." (356d)

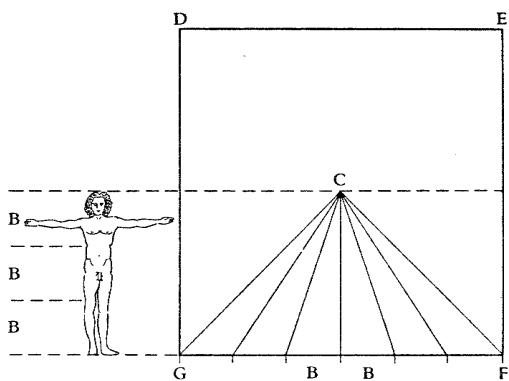
LEON BATTISTA ALBERTI, *De pictura*, 1435. Quoted from the Penguin Classics edition, translated by Martin Kemp, 1991.

"Let us ... start from the opinion of philosophers who say that surfaces are measured by certain rays, ministers of vision as it were, which they therefore call visual rays, since by their agency the images of things are impressed on the senses. ... Let us imagine the rays, like extended very fine threads gathered tightly in a bunch at one end, going back together inside the eye where lies the sense of sight." (§5) "The eye measures [the size of objects] with the extrinsic rays rather like a pair of dividers." (§6) "The extrinsic rays, which hold on like teeth to the whole of the outline [of a body], form an enclosure around the entire surface like a cage. This is why they say that vision takes place by means of a pyramid of rays." (§7)

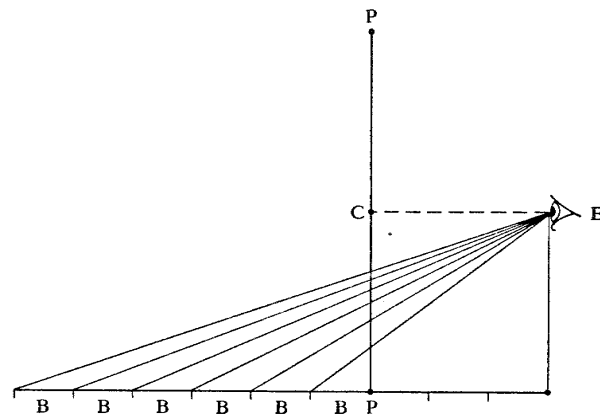
"A painting will be the intersection of a visual pyramid ..., represented by art with lines and colours on a given surface." (§12) "To do this well, I believe nothing more convenient can be found than the veil, ... whose usage I was the first to discover. It is like this: a veil loosely woven of fine thread ... divided up by thicker threads into as many parallel square sections as you like, and stretched on a frame. I set this up between the eye and the object to be represented, so that the visual pyramid passes through the loose weave of the veil. ... [In this way] the boundaries of the surfaces can easily be established accurately on the painting panel; for just as you see the forehead in one parallel, the nose in the next, ... and everything else in its particular place, so you can situate precisely all the features on the panel or wall which you have similarly divides into appropriate parallels." (§31)



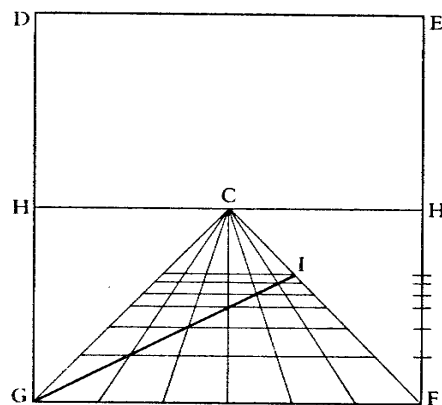
“Let me tell you what I do when I am painting. ... I decide how large I wish the human figures in the painting to be. I divide the height of this man into three parts. ... With this measure I divide the bottom line ... into as many parts as it will hold. Then I establish a point ... wherever I wish; and as it occupied the place where the centric ray strikes, I shall call this the centric point. The suitable position for this centric point is no higher from the base line than the height of the man ... for in this way both the viewers and the objects in the painting will seem to be on the same plane. Having placed the centric point, I draw lines from it to each of the divisions on the base line.” (§19)



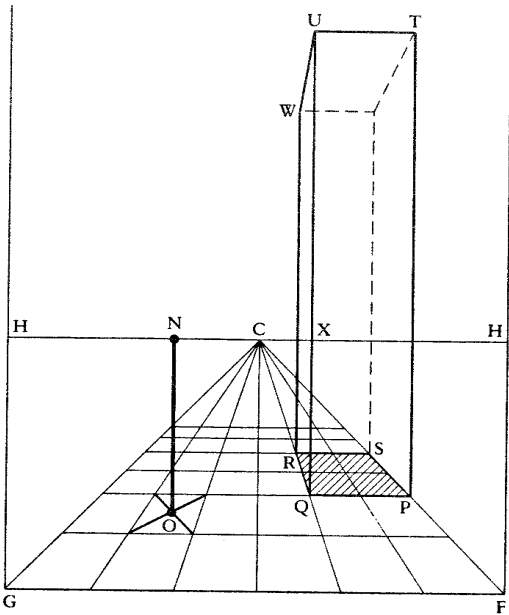
“As regards the successive [parallels of a tiled floor] I observe the following method. I have a drawing surface [on the canvas margin] on which I describe a single straight line, and this I divide in parts like those into which the base line of the rectangle is divided. Then I place a point above this line, directly over one end of it, at the same height as the centric point ... and from this point I draw lines to each of the divisions of the line. Then I determine the distance I want between the eye of the spectator and the painting, and, having established the position of the intersection at this distance, I effect the intersection with ... a perpendicular. ... This perpendicular will give me, at the places it cuts the other lines, the measure of what the distance should be in each case between the transverse equidistant lines of the pavement.” (§20)



“A proof of whether they are correctly drawn will be if a single straight line forms the diagonal of connected quadrangles in the pavement. ... When I have carefully done these things, I draw a line across, ... which ... passes through the centric point. This line is for me a limit or boundary, which no quantity exceeds that is not higher than the eye of the spectator ... This is why men depicted standing in the parallel [to the horizon] furthest away are a great deal smaller than those in the nearer ones—a phenomenon which is clearly demonstrated by nature herself, for in churches we see the heads of men walking about, moving at more or less the same height, while the feet of those further away may correspond to the knee-level of those in front.” (§20)



“On the pavement that is divided up into parallels, you have to construct the sides of walls and other similar surfaces. ... I will explain briefly how I proceed in this construction. I begin first from the foundations. I draw the breadth and length of the walls on the pavement. ... I determine what I wish their length and breadth to be by the parallels traced on the pavement, for I take up as many parallels as I want them to be braccia. ... So, from the scale of the parallels I easily draw the width and length of walls that rise from the ground. Then I go from there without any difficulty to do the heights of the surfaces. ... [For example,] if you want this [height] from the ground to the top to be [three] times the height of a man ..., you must continue it upwards [two] times again the distance from the centric line to the foot of the [wall].” (§33)



SAMUEL Y. EDGERTON, *Renaissance Rediscovery of Linear Perspective*, Icon, 1976.

"Perspective rules were accepted by ... early artists because it gave their depicted scenes a sense of harmony with natural law, thereby underscoring man's moral responsibility within God's geometrically ordered universe." A passage in Dante "upholds the wisdom of looking someone 'straight in the eye', with all its overtones of detecting another's truthfulness and communicating one's own sincerity"—perhaps this is why Renaissance artists preferred the central viewpoint where walls and floor tiles tend towards the centric point in the middle of the horizon. "During the fourteenth century there had been a temporary favoring of charming oblique views. Thereafter it was as if painters wanted to recapture the solemn spirit of the old traditional Christian messages" and "came more and more to believe that things planned or seen from a central viewpoint had greater monumentality and moral authority."

CLIFFORD D. CONNER, *A People's History of Science: Miners, Midwives, and Low Mechanics*, Nation Books, 2005.

"The invention of perspective by the Renaissance artists, ... by demonstrating that mathematics could be usefully applied to physical space itself, [constituted] a momentous step ... toward the general representation of physical phenomena in mathematical terms." (270)

DAVID WOOTTON, *The Invention of Science: A New History of the Scientific Revolution*, Allen Lane, Penguin Random House, 2015.

"The mathematization of the sublunary world begins not with Galileo but with Alberti." (§5.8)

H. FLORIS COHEN, *The Scientific Revolution: A Historiographical Inquiry*, University of Chicago Press, 1994.

"Galileo may fruitfully be seen as the culmination point of a tradition in Archimedean thought which, by itself, had run into a dead end. What enabled Galileo to overcome its limitations ... seems easily explicable upon considering Galileo's background in the arts and crafts." (349)

11.3. ☞ In the early modern period, who had respect for mathematics, and why? Who were critical of mathematics, and why?

11.4. ☞ What was the relation between mathematics and religion?

11.5. ☞ What was the relation between mathematics and science?

11.6. ☞ Which themes in these debates are timeless, and which were only relevant in their particular context?

11.7. Additional reading: "When pirates studied Euclid" (<https://aeon.co/essays/how-european-sailors-learned-> Which of the following uses of mathematical methods are mentioned in the article?

- ☐ determine position by observation of the night sky
- ☐ determine time by observation of the night sky
- ☐ use geometry for carpentry-related problems on the ship
- ☐ determine the course needed to catch another ship
- ☐ determine when high and low tides will occur
- ☐ determine the area of a sail suitable for given requirements (such as the weight of the ship etc.)
- ☐ determine the angle of travel given the angle of the wind and the sail

OTTO GEORG VON SIMSON, *The Gothic Cathedral*, Princeton University Press, 1988.

"At least one literary document survives that explains the use of geometry in Gothic architecture: the minutes of the architectural conferences held during 1391 and the following years in Milan. ... The question debated at Milan is not whether the cathedral is to be built according to a geometrical formula, but merely whether the figure to be used is to be the square ... or the equilateral triangle. ... The minutes of one particularly stormy session relate an angry dispute between the French expert, Jean Mignot, and the Italians. Overruled by

them on a technical issue, Mignot remarks bitterly that his opponents have set aside the rules of geometry by alleging science to be one thing and art another. Art, however, he concludes, is nothing without science, *ars sine scientia nihil est*. ... This argument was considered unassailable even by Mignot's opponents. They hasten to affirm that they are in complete agreement as regards this theoretical point and have nothing but contempt for an architect who presumes to ignore the dictates of geometry."

ROGER ASCHAM, *The Schoolmaster*, 1570.

"Some wits, moderate enough by nature, be many times marred by over much study and use of some sciences, namely, music, arithmetic, and geometry. These sciences, as they sharpen men's wits over much, so they charge men's manners over sore, if they be not moderately mingled, and wisely applied to some good use of life. Mark all mathematical heads, which be wholly and only bent to those sciences, how solitary they be themselves, how unapt to serve in the world."

WILLIAM KEMPE, translator's dedication in Ramus, *The Art of Arithmetick*, 1592.

"Take away arithmetic, ye take away the merchant's eye, whereby he seeth his direction in buying and selling; ye take the goldsmith's discretion, whereby he mixeth his metals in due quantities; ye take away the captain's dexteritie, whereby he embattaileth his army in convenient order; finally ye take from all sorts of men, the faculty of executing their functions aright. Arithmetic then teacheth unto us matters in divinity, judgeth civil causes uprightly, cureth diseases, searcheth out the nature of things created, singeth sweetly, buyeth, selleth, maketh accompts, weigheth metals and worketh them, skirmisheth with the enemy, goeth on warfare, and setteth her hand almost to every good work, so profitable is she to mankind."

TOBIAS DANTZIG, *Number: The Language of Science*, Masterpiece Science edition, Pi Press, 2005.

"There is a story of a German merchant of the fifteenth century, which I have not succeeded in authenticating, but it is so characteristic of the situation then existing that I cannot resist the temptation of telling it. It appears that the merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy, which in his opinion was the only country where such advanced instruction could be obtained." (26)

CHRISTOPH J. SCRIBA, *The Autobiography of John Wallis, F.R.S., Notes and Records of the Royal Society of London*, 25(1), 1970, 17–46.

When Wallis went to Oxford in 1632 there was no one at the university who could teach him mathematics. "For Mathematicks, (at that time, with us) were scarce looked upon as Academical studies, but rather Mechanical; as the business of Traders, Merchants, Seamen, Carpenters, Surveyors of Lands, or the like." (27)

CHRISTOPHER CLAVIUS, *In disciplinas mathematicas prolegomena*, 1574. Translation quoted from J. M. Lattis, *Between Copernicus and Galileo*, University of Chicago Press, 1994, 35–36.

"The theorems of Euclid and the rest of the mathematicians, still today many years past, retain ... their true purity, their real certitude, and their strong and firm demonstrations. ... And thus so much do the mathematical disciplines desire, esteem, and foster truth that they reject not only whatever is false, but even anything merely probable ... So there can be no doubt but that the first place among the other sciences should be conceded to mathematics." Mathematics is "not only useful, but in fact necessary" in many fields, but "of all these benefits [of mathematical studies], perhaps the greatest is the entertainment and pleasure that fills the soul as a result of the cultivation and exercise of those arts."

THOMAS HOBBS, *De Cive*, 1642. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume II.

"And truly the geometricians have very admirably performed their part. For whatsoever assistance doth accrue to the life of man, whether from the observation of the heavens, or from the description of the earth, from the notation of times, or from the remotest experiments of navigation; finally, whatsoever things they are in which this present age doth differ from the rude simpleness of antiquity, we must acknowledge to be a debt which we owe merely to geometry. If the moral philosophers had as happily discharged their duty, I know not what could have been added by humane Industry to the completion of that happiness, which is consistent with humane life." (iv)

AMIR ALEXANDER, *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World*, Scientific American / Farrar, Straus and Giroux, 2014.

—Jesuit opposition to infinitesimals.

The Jesuits were the intellectual leaders of the Catholic world in the 17th century. They ran hundreds of colleges across Eu-

rope, notable as much for their “sheer educational quality” (46) as for their doctrinal role “in the fight to defeat Protestantism” (41).

The Jesuit colleges placed great emphasis on Euclidean mathematics. “It was a logical sequence of studies, but for [the Jesuits] it also represented a deeper ideological commitment. Geometry, being rigorous and hierarchical, was, to the Jesuit, the ideal science. The mathematical sciences that followed—astronomy, geography, perspective, music—were all derived from the truths of geometry ... Consequently, Clavius’s mathematical curriculum ... demonstrated how absolute eternal truths shape the world and govern it” (72), thereby serving as a model for their religious doctrine and worldview. “Euclidean geometry thus came to be associated with a particular form of social and political organization, which ... the Jesuits strived for: rigid, unchanging, hierarchical, and encompassing all aspects of life.” (218)

For this reason, “the Jesuits reacted with ... fury to the rise of infinitesimal methods. For the mathematics of the infinitely small was everything that Euclidean geometry was not. Where geometry began with clear universal principles, the new methods began with a vague and unreliable intuition that objects were made of a multitude of minuscule parts. Most devastatingly, whereas the truths of geometry were incontestable, the results of the method of indivisibles were anything but” (175), thereby undermining “the Jesuit quest for a single, authorized, and universally accepted truth” (176).

Thus infinitesimal mathematics was dangerous to the Jesuits not for intrinsic mathematical reasons but because it was associated with diversity of thought unchecked by authority. “‘Unless mind are contained within certain limits’, warned Father Leone Santi, prefect of studies at the Collegio Romano ... ‘their excursions into exotic and new doctrines will be infinite’, leading to ‘great confusion and perturbation to the Church.’” (122)

Consequently, “In a fierce decades-long campaign, the Jesuits worked relentlessly to discredit the doctrine of the infinitely small and deprive its adherents of standing and voice in the mathematical community. Their efforts were not in vain: as 1647 was drawing to a close, the brilliant tradition of Italian mathematics was coming to an end as well.” (117)

According to Alexander the Jesuit campaign against infinitesimals was extremely successful and influential. As he puts it, “champions of the infinitely small (Galileo, Cavalieri, and Torricelli) pioneered new techniques that would transform the very foundations of mathematical inquiry and practice. But when the Jesuits triumphed over the advocates of the infinitely small, this brilliant tradition died a quick death,” (178), leaving “no one left in Italy to carry the torch of the infinitely small” (165). “In Italy, the stage was set for centuries of backwardness and stagnation.” (180) “No city or prince wished to risk the wrath of the Jesuits, and as a result, no university chairs or positions of honor at princely courts were in the offing for supporters of the infinitely small.” (165)

It should be noted, however, that Cavalieri’s generation did in fact have a number of direct followers in prominent university

positions. Cavalieri’s student “Pietro Mengoli (1626-84) succeeded Cavalieri to the mathematics chair at Bologna,” (164) a position he held for the remaining 39 years of his life. Angeli was another student of Cavalieri’s, and a *prima facie* counterexample to Alexander’s thesis. “In 1662, he was appointed to the chair of mathematics at the University of Padua, a position once held by Galileo. The Jesuits, so powerful elsewhere in Italy, could only fume as the upstart ... was raised to one of the most prestigious mathematical posts in all Europe.” (170-171) “Angeli ... took on the Jesuits like no one had dared since the days of Galileo himself. He called them names [and] ridiculed [them],” (170) and “published no fewer than nine books promoting and using the method of indivisibles” (174), and held his chair for the rest of his life.

—Mathematics and politics in Britain.

Wallis’s work on infinite series was based on daring, unrigorous extrapolations and generalisations, which he considered “‘a very good Method of Investigation ... which doth very often lead us to the early discovery of a General Rule’. Most important, ‘it need not ... any further Demonstration.’” (270)

Hobbes, by contrast, appealed to the authority and rigour of Euclidean geometry as a model for reasoning as well as political organisation.

“Wallis and Hobbes both believed that mathematical order was the foundation of the social and political order, but beyond this common assumption, they could agree on practically nothing else. Hobbes advocated a strict and rigorous deductive mathematical method, which was his model for an absolutist, rigid, and hierarchical state. Wallis advocated a modest, tolerant, and consensus-driven mathematics, which was designed to encourage the same qualities in the body politic as a whole.” (256)

Wallis’s vision of mathematics was very agreeable to the experimental scientists of the Royal Society. “Experimentalism is a humbling pursuit, very different from the brilliance and dash of systematic philosophers such as Descartes and Hobbes. It is, wrote Sprat, ‘a laborious philosophy ... that teaches men humility and acquaints them with their own errors’. And that is precisely what the founders of the Royal Society liked about it. Experimentalism, as Sprat noted, ‘removes all haughtiness of mind and swelling imaginations’, teaching men to work hard, to acknowledge their own failures, and to recognize the contributions of others.” (253)

“Mathematics, [the Royal Society founders] believed, was the ally and the tool of the dogmatic philosopher. It was the model for the elaborate systems of the rationalists, and the pride of the mathematicians was the foundation of the pride of Descartes and Hobbes. And just as the dogmatism of those rationalists would lead to intolerance, confrontation, and even civil war, so it was with mathematics. Mathematical results, after all, left no room for competing opinions, discussions, or compromise of the kind cherished by the Royal Society. Mathematical results were produced in private, not in a public demonstration, by a tiny priesthood of professionals who spoke their own language; used their own methods, and ac-

cepted no input from laymen. Once introduced, mathematical results imposed themselves with tyrannical, power, demanding perfect assent and no opposition. This, of course, was precisely what Hobbes so admired about mathematics, but it was also what Boyle and his fellows feared: mathematics, by its very nature, they believed, leads to claims of absolute truth, dogmatism, threats of tyranny.” (256)

JED Z. BUCHWALD & I. BERNARD COHEN (EDS.), *Isaac Newton's Natural Philosophy*, MIT Press, 2004.

The Royal Society in London was long fraught with tension between mathematicians and naturalists. “There has been much canvassing and intrigue made use of, as if the fate of the Kingdome depended on it” (77). “On the eve of Newton's election as president, matters had deteriorated to such an extent that various fellows could be restrained only with difficulty from a public exchange of blows (or, in one case, the drawing of swords)” (93).

So what was this conflict on which “the fate of the Kingdome” depended? The “philomats” identifying with Newton attacked the naturalists thus: “That Great Man [Newton] was sensible, that something more than knowing the Name, the Shape and obvious Qualities of an Insect, a Pebble, a Plant, or a Shell, was requisite to form a Philosopher, even of the lowest rank, much more to qualifie one to sit at the Head of so great and learned a Body.” (77)

The naturalists, for their part, identified with Bacon, who had complained about “the daintiness and pride of mathematicians, who will needs have this science almost domineer over Physic. For it has come to pass, I know not how, that Mathematics and Logic, which ought to be but the handmaids of Physic, nevertheless presume on the strength of the certainty which they possess to exercise dominion over it.” (80)

Similar points were raised many times, as here in 1700 by a minor figure: “Mathematical Arguments, of which the World is become most immoderately fond, looking upon every thing as trivial, that bears no relation to the Compasse, and establishing the most distant parts of Humane Knowledge; all Speculations, whether Physical, Logical, Ethical, Political, or any other upon the particular results of number and Magnitude. ... In any other commonwealth but that of Learning such attempts towards an absolute monarchy would quickly meet with opposition. It may be a kind of treason, perhaps, to intimate thus much; but who can any longer forbear, when he sees the most noble, and most usefull portions of Philosophy lie fallow and deserted for opportunities of learning how to prove the Whole bigger than the Part, etc.” (90)

HELENA PYCIOR, *Symbols, Impossible Numbers, and Geometric Entanglements*, Cambridge University Press, 1997.

In the 17th century, some took “the symbolical style [of algebra]

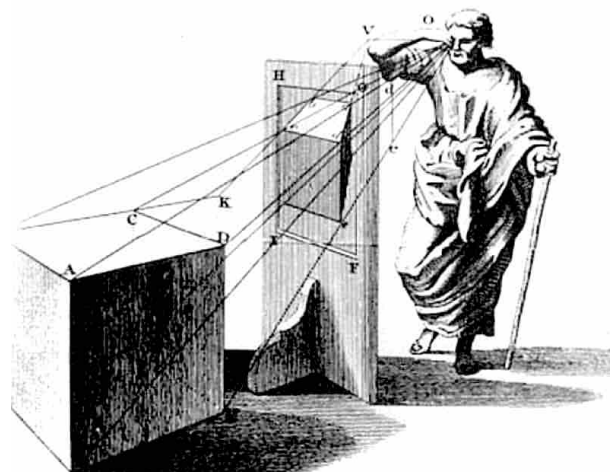
as a model for terse, scientific expression.” Indeed, the Royal Society “exacted from all their members a close, naked, natural way of speaking ... bringing all things as near the Mathematicall plainness as they can.” (Sprat, 46) Others disagreed. “Symbols are poor unhandsome, though necessary, scaffolds of demonstration; and ought no more to appear in public, than the most deformed necessary business which you do in your chambers.” (Hobbes, 145) Pages of algebra often look “as if a hen had been scraping there.” (Hobbes, 147)

Negative numbers. “That which most perplexes narrow minds in this way of thinking, is, that in common life, most quantities lose their names when they cease to be affirmative, and acquire new ones so soon as they begin to be negative: thus we call negative goods, debts; negative gain, loss; negative heat, cold; negative descent, ascent, &c.: and in this sense indeed, it may not be so easy to conceive, how a quantity can be less than nothing, that is, how a quantity under any particular denomination, can be said to be less than nothing, so long as it retains that denomination.” (Saunderson, 287)

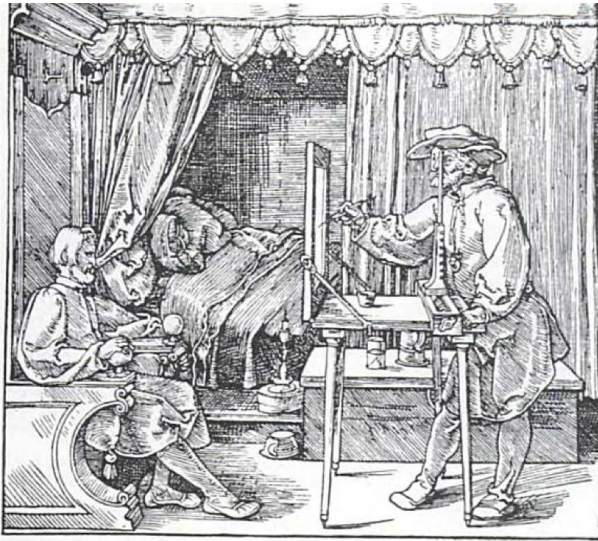
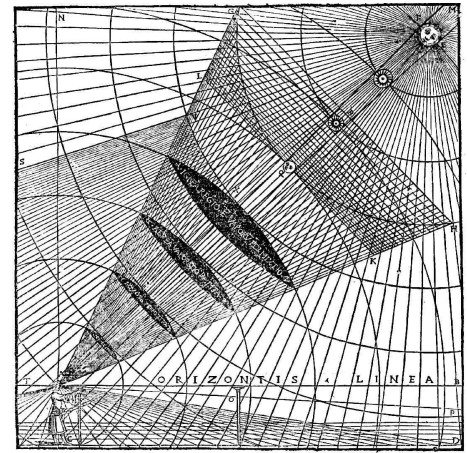
Technology in teaching. “That the true way of Art is not by Instruments, but by demonstration: and that it is a preposterous course of vulgar Teachers, to beginne with Instruments, and not with the Sciences, and so in stead of Artists, to make their Schollers onely doers of tricks, and as it were jugglers.” (Oughtred, 68)

§ 12. Perspective art

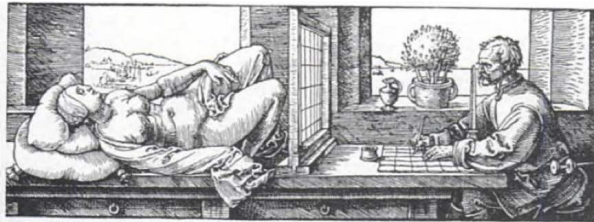
Perspective painting is based on this principle:



That is to say, each point that is to be drawn is connected with a line to the observer's eye, and the intersection of this line with the canvas marks where the point is depicted. This principle was discovered in the Renaissance. One of its early advocates was Albrecht Dürer, who illustrated it as follows.

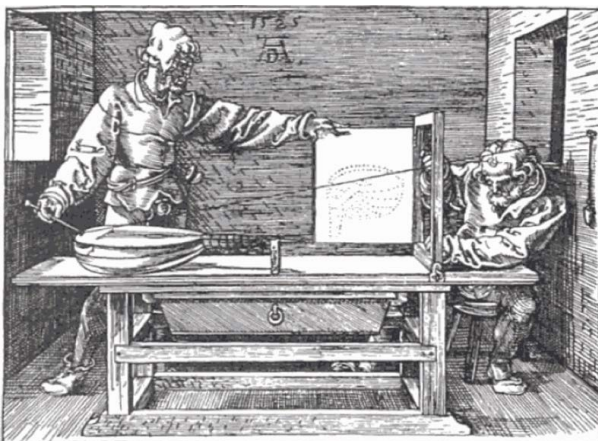


(c) Parallel lines meet at the horizon.



12.2. You see a vacation photo of a person standing on a beach with the ocean behind her. The horizon is aligned with the top of her head. How tall was the person who took the photo?

- ☐ Taller than the subject.
- ☐ As tall as the subject.
- ☐ Shorter than the subject.



12.3. This is Mantegna's *St. James led to Execution* (1452):



Mantegna utilised the principles of perspective to great effect in this work. Explain.

12.1. Prove the following basic facts about perspective painting.

- (a) Parallel lines meet at one point in the picture. Like railroad tracks, for example.
- (b) The horizon is always perpendicularly in front of the observer's eye.

12.4. Show that Duccio's *The Last Supper* (1310) does not obey the laws of perspective:



Duccio instead follows an alternative artistic principle, namely to show each object from its most advantageous viewpoint—the vase is seen from the side, the food plate from above, the ceiling from below—without regard for the impossibility of seeing all those perspectives all at once.

12.5. Discuss how this alternative principle is prominent also in this typical ancient Egyptian painting:



12.6. Which of the following paintings follow correct mathematical perspective?

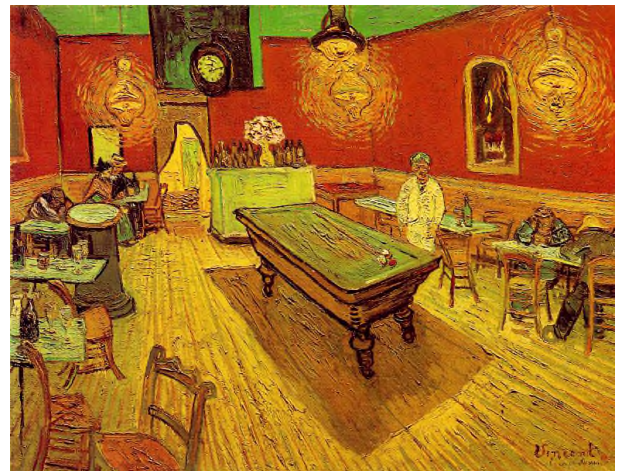
☐ Annunciation, Leonardo Da Vinci



☐ The Miracle of the child falling from the balcony, Simone Martini



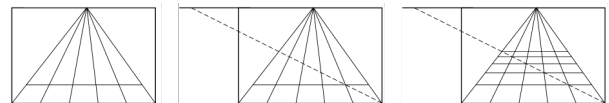
☐ The Night Cafe, Van Gogh



☐ Piazza San Marco, Canaletto



Alberti's *Della pittura* (1435) gives the following construction of the perspective view of a tiled floor:

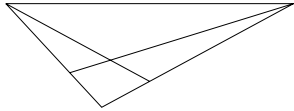


12.7. Explain why this construction gives a true perspective

view of a tiled floor.

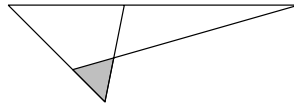
12.8. Can you use the same principles to draw an oblique view of a tiled floor, starting from a figure like the one below?

- ☐ Yes, and it will be the perspective image of congruent and square tiles.
- ☐ Yes, and it will be the perspective image of congruent though not necessarily square tiles.
- ☐ No, this starting configuration does not contain sufficient information to complete the true perspective view of the floor.



12.9. Can you use the same principles to draw an oblique view of a floor tiled with congruent triangles, starting from a figure like the one below?

- ☐ Yes.
- ☐ No.



The principle of tiled floors is useful not only for pictures of floor but also as a guide to sizes generally.

12.10. In the painting “A Woman Drinking with Two Men” by Pieter de Hooch, which of the two standing women is tallest?

- ☐ The one on the left.
- ☐ The one on the right.
- ☐ They are about equally tall.

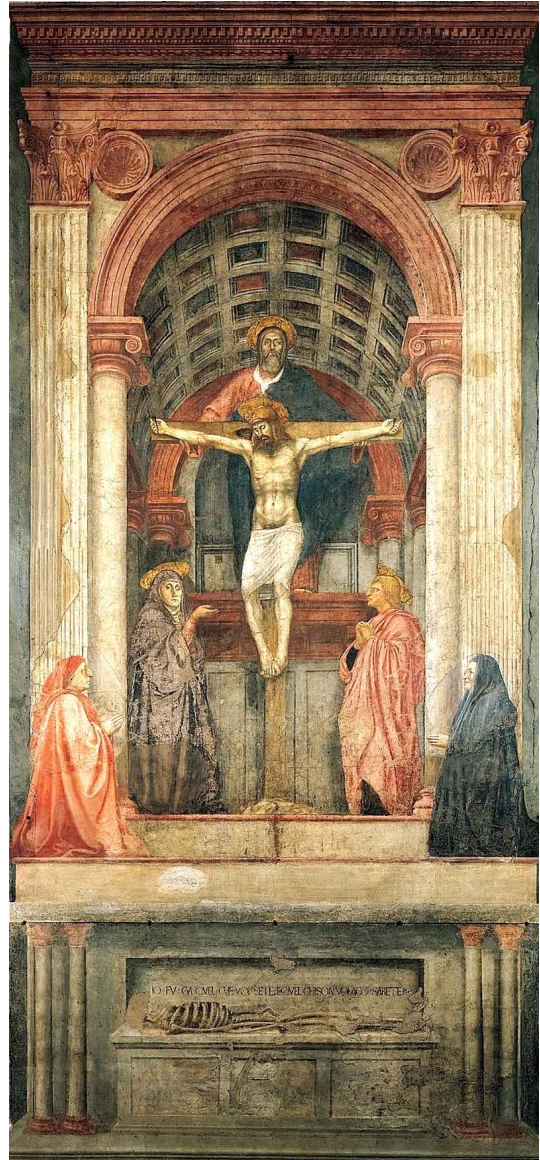


12.11. In this painting there is no tiled floor and no parallel lines. Nevertheless, estimate where the horizon is and introduce your own parallels to check whether the heights

of the figures in the background are in perspective agreement with those in the foreground.



This is Masaccio's “Holy Trinity” (1425–1428):

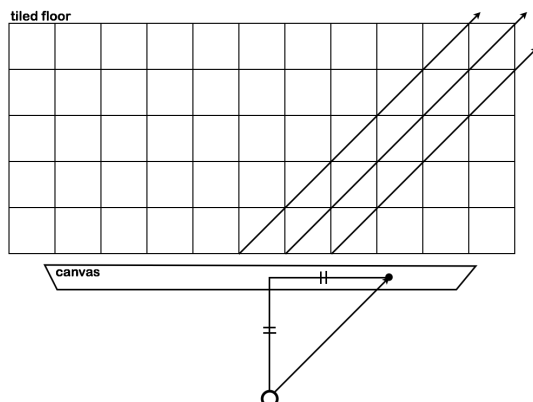


This work is painted directly on the wall of a church in Florence. It is designed to give the illusion of depth, as if the scene was really there in place of the wall. However, this only works for a specific vantage point.

12.12. What is the eye level of the observer in Masaccio's “Holy Trinity”?

- ☐ Level with the heads of the two foremost figures in the painting.
- ☐ Higher.
- ☐ Lower.

12.13. In addition to the height, the ideal viewing distance must also be taken into account. Explain how it can be determined for any painting containing a perspective view of a tiled floor using the following diagram:



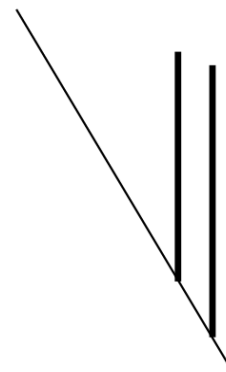
12.14. Can we use this method to estimate the ideal viewing distance for Masaccio's "Holy Trinity" from the tiles in the ceiling?

- ☐ Yes. Mathematically, it doesn't matter whether we are using a curved or a flat tiled surface for the above construction.
- ☐ No. A flat tiled surface is required.
- ☐ Sort of, if we use the top tiles only and treat them as if they were flat.

12.15. When Alberti describes how to draw tiled floors in *De pictura*, does he assume a fixed viewing distance?

- ☐ Yes, following Alberti's recipe the viewing distance is always the same relative to the size of the canvas.
- ☐ Yes, following Alberti's recipe the viewing distance is always the same relative to the size of the human figure used as a size reference in the painting.
- ☐ No, Alberti's description allows for the viewing distance being selected arbitrarily.

12.16. The figure below shows a perspective view of two telephone poles standing on the side of a straight road. Explain how to draw the correct perspective view of the third telephone pole. Justify all your steps. (In reality, of course, the telephone poles are all of the same size and equally spaced.)



§ 13. Projective geometry

The principles of perspective painting also have more mathematical applications, as we shall now see. When studying the theory of projections mathematically we must idealise the painting situation somewhat to include the possibility of seeing through the back of our heads.

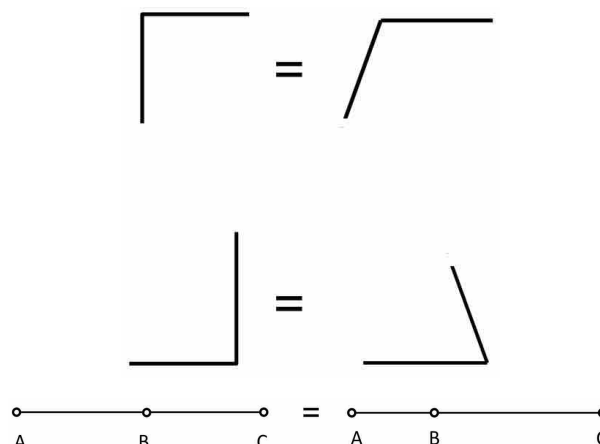
13.1. Argue that if you are painting a plane landscape everything in front of you will appear below the horizon line on the canvas. Then argue that there is a sense in which everything behind you will appear above the horizon, if it is projected onto the canvas following the same rule as the points in front of you.

13.2. What does a painting of railroad tracks look like using this kind of projection?

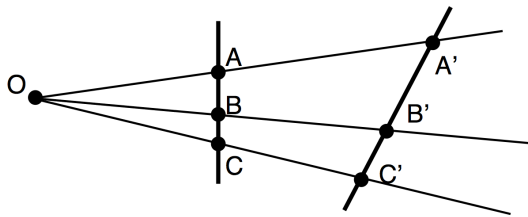
13.3. Consider two points on either side of the railroad. Is there any projection for which the images of these two points can be connected on the canvas without crossing the railroad tracks?

- ☐ Yes
- ☐ No

13.4. Convince yourself that the following figures can be projected onto one another. Hint: you can literally "see" this by having one of the two objects involved printed on a transparency sheet and aligning it with the other visually.



One-dimensional projective geometry concerns the projections of lines onto lines, such as this:



13.5. Can the first configuration below be projected onto the second? (You are free to move the lines and choose the point O of origin of the projection.)

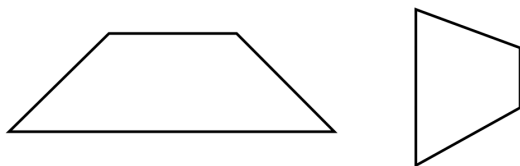


☐ Yes

☐ No

For a projection of a plane onto a plane, four points rather than three are required to determine the projection. This can be seen from the fact that squares often appear in almost any quadrilateral shape.

13.6. In which everyday situations do you see squares looking like those below?



13.7. But can a square ever look like this:



Prove that it can. (Hint: place O at infinity.)

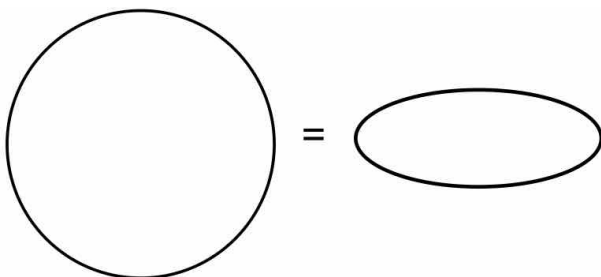
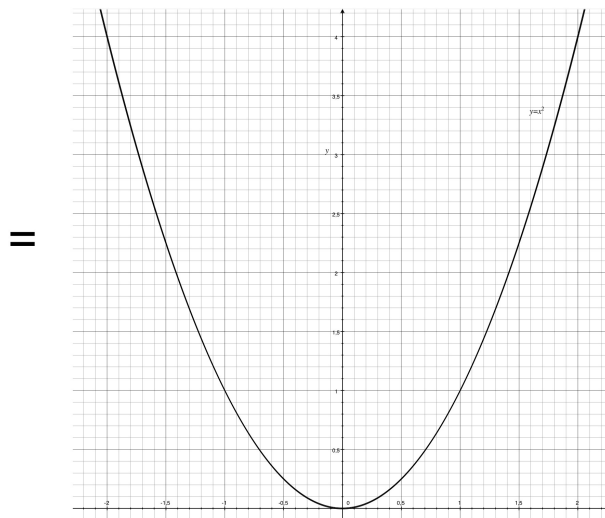
13.8. Can $\odot$ be projected onto $\oslash$?

☐ Yes

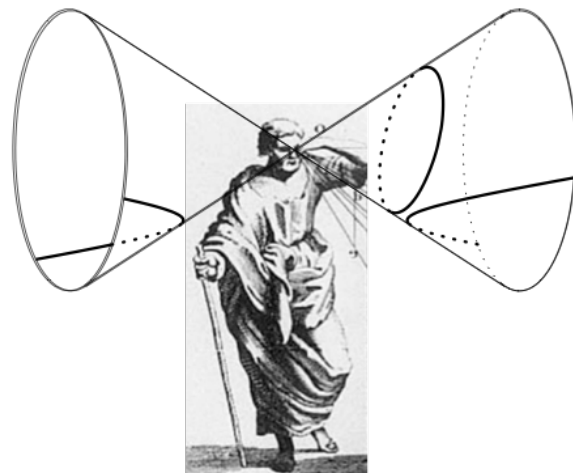
☐ No

13.9. Show that, in a perspective painting, the image of a circle is always a conic section (defined in §14).

13.10. Use a circle printed on a transparency sheet to convince yourself of these projective equivalencies:



13.11. Show more generally that any conic section can be the perspective view of any other. Hint:



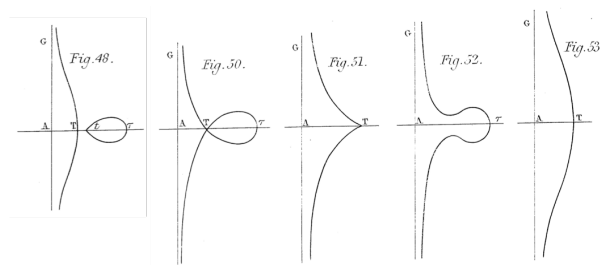
13.12. Argue that the three types of conic sections differ in how many “points at infinity” they have (i.e., number of intersections with the horizon).

The *degree* of a curve means the greatest number of intersections it can have with a line.

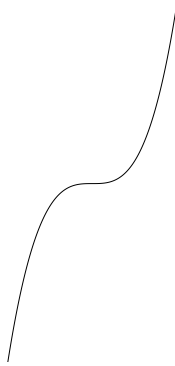
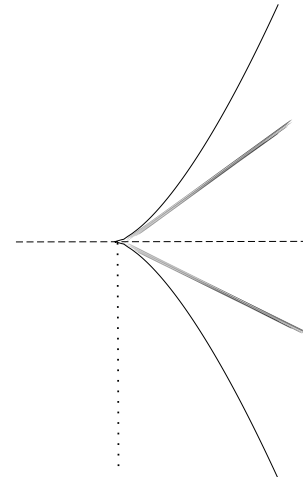
13.13. Of what degree are conic sections?

13.14. Argue that the degree of a curve is preserved under projection.

The idea thus suggests itself to study the projective classification of curves of degree three, or cubic curves. Newton did this and found that there are seven “species,” as he called them:



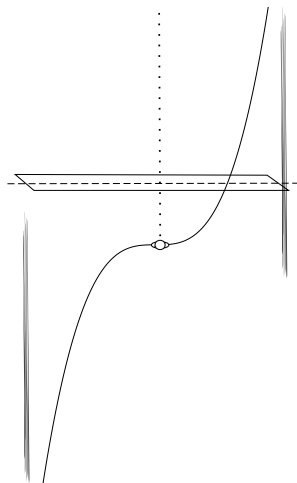
This is indeed the zoology of mathematics, but where is the prototypical cubic curve, $y = x^3$, which we all know looks like this:



13.15. This curve is in fact projectively equivalent to Newton's

- ☐ first
- ☐ second
- ☐ third
- ☐ fourth
- ☐ fifth

Hint: Imagine standing on top of $y = x^3$ with the canvas in front of you like this:



Then try to see why the painting comes out like this:

13.16. Find an image of the 1,500-year-old Archway of Ctesiphon, or Taq Kasra, in Iraq. Is it parabolic? How can you tell?

Projective geometry suggests a new way of conceptualising what geometry is. Felix Klein articulated this view of geometry in his influential Erlanger Programm of 1872. This point of view says: a geometry is characterised by its group of transformations. In other words: the theorems of a geometry is precisely the set of all properties that are invariant under certain transformations. It is difficult to see the point of this if we think about Euclidean geometry only, but when we start conceptualising projective geometry as another kind of geometry we can appreciate how Felix Klein's point of view gives us a whole new vantage point that synthesises different geometries into a satisfying general picture.

Ordinary Euclidean geometry consists of all theorems that do not change when you apply "isometries"—that is, when you move the paper around in such a way that no distances between points change. If you prove something in Euclidean geometry—let's say the Pythagorean Theorem for a certain triangle drawn on a piece of paper—and then rotate the paper, or slide it over to the other end of the table, the clearly the proof still remains valid. Such transformations are isometries. On the other hand there are transformations that change distances, such as if you tear the paper, or if the "paper" was actually a sheet of rubber that can be stretched or deformed. Of course, Euclidean theorems you proved before such a transformation do not remain valid for the deformed figure you are left with after the transformation has been applied. This is such an obvious point that we never paid attention to it before. But it turns out to be a useful perspective for generalising our concept of geometry.

Just as Euclidean geometry is about whatever is preserved by isometries, projective geometry is about whatever is preserved by projections. In Euclidean geometry we consider two figures to be "the same" if we can put one on top of the other by moving the paper. In projective geometry two figures are "the same" whenever one can be projected onto the other. Many concepts that are central in Euclidean geometry completely "collapse" in projective geometry. A circle is the same thing as an ellipse. An

obtuse angle is the same thing as an acute angle. One meter is the same thing as ten meters. And so on. Yet not everything is the same as everything else. Lines are different from conic sections, and conic sections are different from curves of degree 3, and so on.

Hint: Draw a line from the house outwards. What can you say about the number of crossings?

§ 14. Conic sections

13.17. Justify these claims based on our investigations in §13.

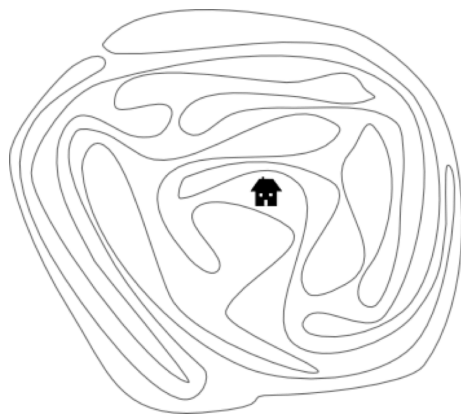
We can go further and allow a more radical group of transformations still. In topology, we study theorems that are invariant under any continuous transformation—we can think of it as rubber band transformations. Two curves are “the same” if a rubber band placed on one can be deformed into the other. Now even the distinction between the degrees of curves collapses.

<u>topology</u>	equivalence relation: deforming rubber band	not preserved: degree of curve
<u>projective geometry</u>	equivalence relation: projection	not preserved: lengths, angles, orientation, point order
<u>Euclidean geometry</u>	equivalence relation: moving the paper	preserved: degree of curve (# intersections with line)

You may think that we went too far with this notion of “sameness.” If “everything is the same as everything else,” then there is no geometry left to speak of; we can’t prove any theorems at all. But in fact the collapse is not as complete as all that. Interesting properties of curves are preserved even in “rubber band geometry.” One aspect of a curve that remains meaningful is that of “inside” and “outside.”

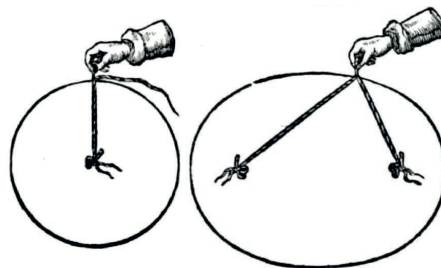
13.18. Is the house inside or outside the region enclosed by the curve?

- ☐ inside
☐ outside

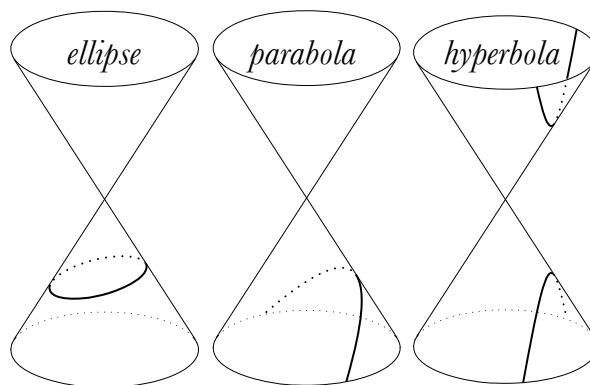


13.19. Find a systematic way—a theorem of topology—to answer questions like this for any picture of this kind.

The string construction of a circle soon suggests this generalization:



The resulting curve in an ellipse, one of the conic sections:



The terms ellipse, parabola, hyperbola mean roughly too little, just right, and too much, respectively.

14.1. Discuss how this relates to the meaning of the words ellipsis (the omission from speech or writing of words that are superfluous; also the the typographical character “...”), parable (a simple story used to illustrate a moral or spiritual lesson), hyperbole (exaggerated statements or claims not meant to be taken literally).

14.2. Match the situation with the corresponding type of conic.

- ☐ the shadow cast on a wall by the lampshade of a lamp; the water jet from the Manneken Pis statue in Brussels; the shape of a hanging chain; the orbit of Halley’s Comet
- ☐ ellipse; parabola; hyperbola; none of the above

The following was another important reason for ancient man to study conic sections.

14.3. (a) “Gnomon” is a fancy word for a stick standing in the ground. The tip of its shadow traces a curve as the sun moves. What type of curve is it? Curves 3 and 6 in the figure below, for example, are both

- ☐ hyperbolas

- ☐ parabolas
- ☐ ellipses

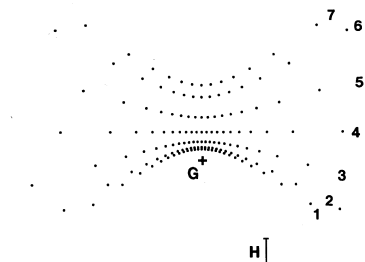


FIGURE 1.34. A sequence of shadow plots made at Seattle. On each plot, the points are half an hour apart. 1, June 22; 2, May 21 or July 23; 3, April 20 or August 24; 4, March 21 or September 23; 5, February 19 or October 24; 6, January 20 or November 23; 7, December 22. G labels the point at which the gnomon was set up perpendicular to the plane of the diagram. The line H shows the height of the gnomon that was used.

(Figure from Evans, *History and Practice of Ancient Astronomy*.)

- (b) How can you find north by using the stick and the curve?
- (c) Suppose the shadow cast by the gnomon in the course of a day is a straight line. What can you conclude from this?
 - ☐ North cannot be determined from shadow measurements that day.
 - ☐ Day and night are equally long.
 - ☐ The sun is perpendicularly above the gnomon at mid day.
 - ☐ Our location is not at the equator of the earth.
 - ☐ The earth is no longer orbiting the sun but is travelling in a straight line toward or away from it.

14.4. Prove by the following steps that the radius and tangent of a circle meet at right angles.

- (a) Explain how we know that the circle has a tangent line at any point in the first place. By tangent line we mean a line that touches the curve without intersecting it. Etymologically, the term tangent comes from the Latin term for touch, as does the name of the dance tango.
- (b) Draw the radius to the point of tangency and reflect the midpoint in the tangent line. What does our theorem mean in terms of the resulting figure? Prove that this is so.

14.5. Let us consider the analogous situation for the case of an ellipse. An ellipse can be defined as the curve for which the sum of the distances to two fixed point are equal, as the string construction shown above suggests. In other words, the distance $F_1B + BF_2$ is the same for any point

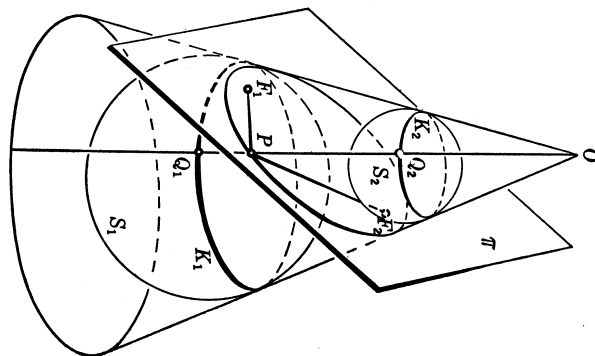
B on the ellipse, where F_1 and F_2 are the endpoints where the string is fastened. We now wish to prove that any tangent to the ellipse forms equal angles with the focal radii. Thus if light or sound is emitted from one focus it will be reflected towards the other.

- (a) Explain how this is a generalisation of the above result for circles.

Reflect F_2 in the tangent, and call the image F'_2 . The straight line $F_1F'_2$ is the shortest distance between these two points. It intersects the tangent in some point B_1 .

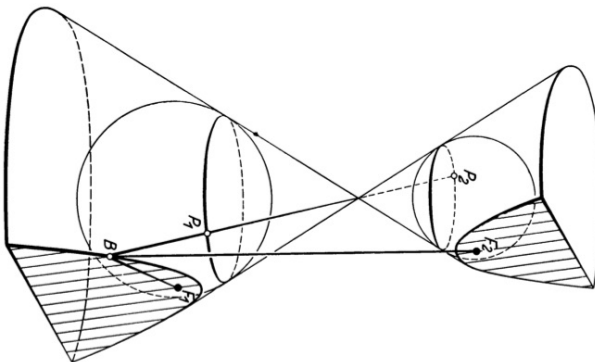
- (b) Argue that B_1 must be B , the point on the ellipse.
- (c) Conclude the proof.

14.6. We have defined ellipses in two ways: as sections of a cone, and in terms of strings or distances from two focal points. How do we know that these two definitions amount to the same thing? Prove it based on this figure.



(Figure from Courant & Robbins, *What is Mathematics?*)

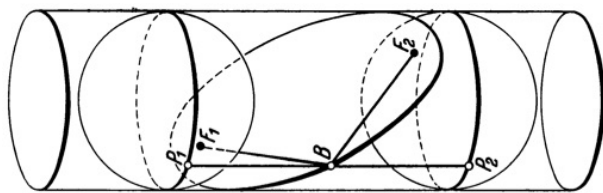
14.7. Adapt the argument for the case of a hyperbolic section of a cone:



(Figure from Hilbert & Cohn-Vossen, *Geometry and the Imagination*.)

14.8. If you cut a cylinder by a plane, is the cross-section an ellipse?

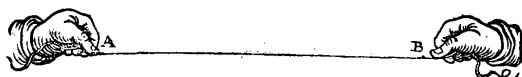
- ☐ yes
- ☐ no



(Figure from Hilbert & Cohn-Vossen, *Geometry and the Imagination*.)

§ 15. Geometry on surfaces

What is straightness? Is a straight line the same thing as the shortest distance between two points? Or to put it in more physical terms: the path of a stretched string.



To get to the bottom of the notion of straightness it is useful to consider not only the usual plane but also other surfaces. Consider:

15.1. What are the straight lines on a cylinder?

Here, “stretched string” and “shortest distance” are not the same thing. It is arguably the stretched string that gets it right. We feel that straightness is a “local” property. We can alter the distance characterisation of straightness to reflect this. Definition: a curve is a locally shortest path if, for any given point P on the curve, there is a neighborhood around that point such that the distance along the curve between any two points on the curve in that neighborhood is the shortest possible distance between those points.

15.2. Are locally shortest paths the same thing as stretched-string paths on the cylinder? In a flat plane?

To appreciate the geometry of a surface—its “intrinsic” geometry, as we say—we should forget for a moment that it is located in three-dimensional space. We should look at it through the eyes of a little bug who crawls around on it and thinks about its geometry but who cannot leave and is unaware of any other space beyond this surface. Think of for example those little water striders that you see running across the surfaces of ponds. They know the surface of the pond ever so well. They can feel any little movement on it. But they are quite oblivious to the existence of a third dimension outside of their surface world. This makes the water strider an easy prey for a bird or a fish that strikes it without first upsetting the surface of the water.

Thinking about the intrinsic geometry of surfaces in this way forces us to realise that what we often take for granted as “obvious” objective truths in geometry are really a lot more specific to our mental constitution and unconscious assumptions than we realise. In some ways we are as ignorant of our own limitations as the water strider.

Instead of using stretched strings, straight lines can also be defined as curves possessing half-turn symmetry about every point: a curve has half-turn symmetry if, for any given point

P on the curve, there is a neighbourhood around that point such that when this neighbourhood is rotated about P by half the angle-measure around P the curve ends up on top of itself. More loosely, a curve is straight if it always “cuts angles in half.” To test for this kind of straightness on surfaces one can use the “ribbon test”: if a ribbon or band can be laid flatly along the curve without creasing on either side, then the curve is straight.

15.3. Are half-turn-symmetry lines the same thing as stretched-string lines on the cylinder?

15.4. For each of the following real-world phenomena related to straightness, try to pinpoint in mathematical terms what aspect of straightness is involved.

- The edge you create when you fold a piece of paper.
- The path of a cross-Atlantic flight.
- The axis of rotation when a solid body is rotated. (Such as a döner spit or a basketball spinning on your finger tip.)
- The heart beats through the contraction of muscular threads across its surface.
- Bandaging an injured limb.
- Light rays. But also how your finger looks “broken” when you put half of it in water.
- Mirrors are made flat by rubbing two of them against each other face-to-face, using fine sand or other polishing agent.
- Rowing a boat and applying equal or different amount of force to each oar.

15.5. According to Russo, the logically poor definition of straightness found in the *Elements* was inserted because:

- ☐ It is intuitively useful.
- ☐ It is constructively useful.
- ☐ It is useful for teaching purposes.
- ☐ Formal mathematics was not yet fully developed.

LUCIO RUSSO, The Definitions of Fundamental Geometric Entities Contained in Book I of Euclid's *Elements*, *Archive for History of Exact Science*, 52 (1998), 195–219.

Definition 4 of Euclid's *Elements* reads: “a straight line is [a line] which lies uniformly in respect to [all] its points.” “Definitions like these are today considered useless and their inclusion in the *Elements* is usually seen as a serious flaw in the *Elements*.” But “the presence of the above definitions in our manuscripts of the *Elements* is ... far from warranting their authenticity, in view of the scant reliability of the textual tradition.” (196) Manuscript evidence suggests that “the original text of Book I of Euclid's *Elements* did contain some

of the definitions,” but that “Euclid did not hesitate in using geometrical terms he had not defined in advance” (such as “circumference”)—a practice “avoided in the Imperial age” when “more definitions were again included in textbooks.” (198) In the works of Archimedes and Apollonius (who “belong to the same scientific tradition” as Euclid) “there is nothing analogous to the pseudo-definitions of fundamental geometrical entities contained in the Elements. The introduction of terms implicitly defined through postulates is instead frequent.” (209–210)

Heron of Alexandria (c. 50) wrote a work devoted to, as he says, “describing and sketching for you as briefly as possible ... the technical terms premised in the elements of geometry” (213). Heron’s description of a straight line begins as follows: “a straight line is [a line] which, uniformly in respect to [all] its points, lies upright and stretched to the utmost towards the ends, such that, given two points, it is the shortest of the lines having them as ends.” “Archimedes ... had in fact assumed that among all lines with the same ends the straight line has the minimum length. It is worth noting that Archimedes’ statement was not a ‘definition’, but [a postulate]. In order to draw a ‘definition’ from Archimedes’ postulate, Heron, however, could not restrict his statement to only one couple of points; he had to require that Archimedes’ property should be verified uniformly in respect to all its points ... Heron’s sentence is therefore completely clear.” (215)

“We know that the obscure scholar who compiled the list of definitions in the form in which they now appear in Book I of the Elements was not a mathematician of any value. We have supposed that he had decided to use as definitions of elementary geometrical entities some excerpts from Heron’s long illustrations. In our case he might have truncated Heron’s first sentence as soon as he could get a syntactically correct sentence, even if empty of mathematical meaning.” (215) The goal in doing so may have been “to get a set of short ‘definitions’ suitable to be learnt by heart in the schools ... If such a list was usually premised to the Elements, it could hardly avoid being eventually confused with Euclid’s text.” (203)

15.6. Argue that the “straight lines” on a sphere are parts of “great circles,” i.e., circles whose midpoint is the center of the sphere. The equator is an example of a great circle on the sphere of the earth.

15.7. Are latitude and longitude lines “straight”?

15.8. Wikipedia says: “Spherical geometry obeys two of Euclid’s postulates: the second postulate (‘to produce [extend] a finite straight line continuously in a straight line’) and the fourth postulate (‘that all right angles are equal to one another’). However, it violates the other three: contrary to the first postulate, there is not a unique shortest route between any two points (antipodal points such as the north and south poles on a spherical globe are counterexamples); contrary to the third postulate, a sphere does not contain circles of arbitrarily great radius;

and contrary to the fifth (parallel) postulate, there is no point through which a line can be drawn that never intersects a given line.” Try to argue that Wikipedia is wrong. For which postulate(s) can this be done on reasonable grounds?

☐ Postulate 1

☐ Postulate 2

☐ Postulate 3

☐ Postulate 4

☐ Postulate 5

15.9. Which of the following are arguably true on a sphere?

☐ Some circles are lines.

☐ Some circles are points.

☐ Some lines are points.

☐ None of the above.

15.10. Playfair’s *Elements of Geometry* (1806) defines a straight line thus: “If two lines are such that they cannot coincide in any two points without coinciding altogether, each of them is called a straight line.” Does this definition single out straight lines and only straight lines on the sphere?

☐ Yes

☐ No

Read Sections 1–5 of *Flatland* by Edwin Abbott, available at <https://www.gutenberg.org/ebooks/201>. This work gives a vivid account of what it is like to live in a surface and not know anything about a third dimension. This mirrors our concern with trying to explain geometrical concepts such as straightness “intrinsically” in terms of the surface itself: we must imagine that a “flatlander” thinks this is the only reasonable way to think of straight lines.

15.11. *Flatland* is also a societal critique. Abbott uses the imaginary country of Flatland to highlight the absurdity and folly of many practices in his own society. In particular, he uses geometry to “explain” (or rather satirise) why:

☐ The working class does not unite and rebel in their own interest.

☐ A husband should treat his wife differently in public than in private.

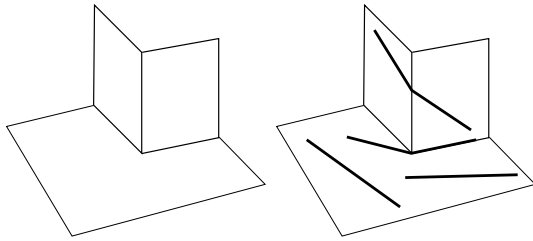
☐ Soldiers are from the low classes.

☐ Modern architecture is of poor taste.

☐ Right hands only must be used for handshakes.

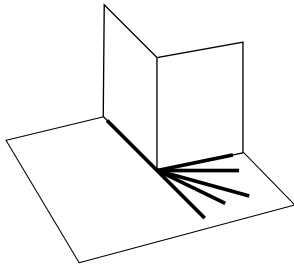
§ 16. 450 degree cone

A “450° cone” looks some thing like a “skyscraper corner”:



You can make one yourself by cutting a slit in an ordinary piece of paper and taping in “90 extra degrees.” The idea is that only the “cone point” is exceptional. All other points of the “450° cone” are indistinguishable from ordinary flat paper. Even the “edges” in the model can be flattened out, so they are not really any different from the flat part of the “walls” or “ground.” The cone point, however is fundamentally different from a point on a flat piece of paper; it is impossible to flatten it out.

Let us investigate how straight lines behave on this surface. Consider for example what happens if we walk along the “wall” and reach the cone point. When we walk along the wall we are certainly going straight, but how should we continue when we reach the corner to keep going straight?



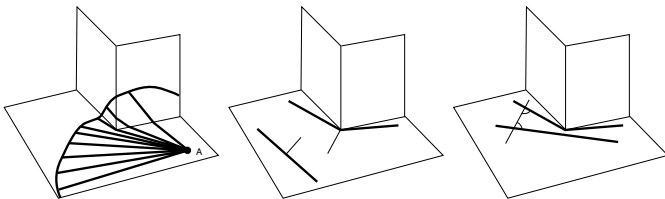
16.1. According to the stretched-string definition of straight lines, which of the lines in the figure are straight?

- ☐ all
- ☐ one
- ☐ none

16.2. According to the half-turn-symmetry definition of straight lines, which of the lines in the figure are straight?

- ☐ all
- ☐ one
- ☐ none

Let us investigate which of Euclid’s postulates hold on this surface. The figures below are hints.



16.3. With the half-turn-symmetry definition of straight lines, which postulates hold on the surface of the 450 degree

cone?

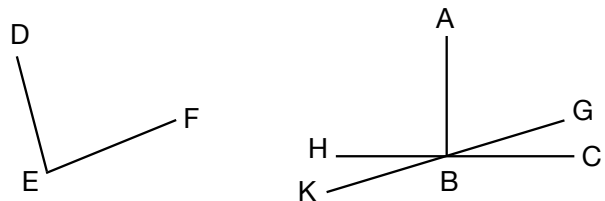
- ☐ Postulate 1
- ☐ Postulate 1’ (Euclid’s Postulate 1 with the added requirement that the line is unique)
- ☐ Postulate 2
- ☐ Postulate 3
- ☐ Postulate 4
- ☐ Postulate 5

16.4. With the stretched-string definition of straight lines, which postulates hold on the surface of the 450 degree cone?

- ☐ Postulate 1
- ☐ Postulate 1’
- ☐ Postulate 2
- ☐ Postulate 3
- ☐ Postulate 4
- ☐ Postulate 5

Proclus tried to prove Postulate 4 as follows:

But if we must provide a graphic proof of this postulate, let us assume two right angles, ABC and DEF . I say they are equal. If they are not, one of them will be greater than the other. Let this be the angle at B . Then if DE be made to coincide with AB , the line EF will fall within the angle, say, at BG . Let BC be extended to H . Since ABC is a right angle, so also is ABH , and the two angles are equal to one another (for a right angle is by definition equal to its adjacent angle). Angle ABH , then, is greater than angle ABG . Now let BG be extended to K . Since ABG is a right angle, its adjacent angle also will be a right angle, equal to ABG . The angle ABK , then, is equal to angle ABG , so that angle ABH is less than ABG . But it is also greater, which is impossible. Hence it is false that a right angle can be greater than a right angle.



16.5. Is Proclus’s proof of Postulate 4 valid?

- ☐ Yes.
- ☐ No, it makes the unwarranted assumption that the adjacent angle of a right angle is itself a right angle.
- ☐ No, it makes the unwarranted assumption that K and G are on opposite sides of HBC .

- ☐ No, it makes the unwarranted assumption that BK is equivalent to the extension of EF.

§ 17. Elements 13–26: angles and congruence

17.1. Postulate 4 is implicitly used in:

- ☐ Proposition 11
☐ Proposition 13
☐ Proposition 14
☐ Proposition 15
☐ None of the above

17.2. (cont.) Is one or more of those propositions false on a 450 degree cone?

- ☐ Yes
☐ No

17.3. Proposition 13. Euclid's first equation is $CBE = CBA + ABE$. His second five-term equation is $DBA + ABC = DBE + EBA + ABC$. The last two terms in that expression are equal to CBE by the first equation. Can Euclid therefore "substitute in" CBE in place of EBA + ABC to get $DBA + ABC = DBE + CBE$, which is precisely what he wants (this is his second to last equation)?

- ☐ Yes, and that is what he does.
☐ Yes, and this would have shortened his proof.
☐ Yes, but this would not have shortened his proof since it needs to apply to both cases mentioned in the statement of the proposition.
☐ No, because the proof needs to apply to both cases mentioned in the statement of the proposition.
☐ No, because there is no Common Notion legitimating such a step.
☐ No, because we cannot assume that $ABE = EBA$.

17.4. Proposition 16. It is assumed that F falls within angle ABC, rather than below (the extension of) BC or above BA. This follows from:

- ☐ BE being an angle bisector.
☐ Two lines cannot enclose a space.
☐ None of the above.

17.5. Proposition 16 does not hold generally on a sphere. Consider the case where B is the south pole, F is the north pole, and AEC is on the equator. If we try to apply Euclid's proof to this configuration, what goes wrong?

- ☐ Nothing. The proof works and the theorem is true for configurations of this type (though not necessarily for all spherical triangles).
☐ The inference based on Proposition 15 fails.
☐ The inference based on Proposition 4 fails.

- ☐ ECD is not greater than ECF.

17.6. Proposition 22. If the condition regarding the lengths of the sides is not satisfied, it is always impossible to construct a triangle from the given sides.

- ☐ Yes
☐ No

17.7. Proposition 22 can be seen as a generalisation of Proposition 1. In our discussion of Proposition 1, we considered some ways of addressing a gap in the reasoning. Which of those ways are also applicable to the analogous gap in the proof of Proposition 22?

- ☐ Carrying out the constructions with ruler and compass.
☐ Allowing conclusions based on what is visually clear in the diagram, as long as it concerns not exact properties but only properties that would still hold even if the diagram was imperfectly drawn.
☐ The notions of "inside" and boundary in Definitions 13-15.

17.8. Proposition 23 shows how to move an angle. But didn't we already assume the ability to move angles in our proof of Proposition 4? (In fact, Proposition 23 is logically dependent on Proposition 4, so we cannot make it the foundation for that part of the proof.)

17.9. Proposition 26. This proof does not "apply" one triangle to the other, as the proofs of Propositions 4 and 8 did. Could this method have been used?

- ☐ Yes, but Euclid's proof is shorter.
☐ Yes, and it would have shortened the proof.
☐ No, it wouldn't work here because we would have to move an angle one of whose associated sides is not given.

17.10. Proposition 26. Euclid has now proved SAS, SSS, and ASA triangle congruence. (Actually his Proposition 26 also includes SAA but I left this out in my edition.) Is there a triangle congruence theorem for every such letter combination? That is, if two triangles have "three things in common" then they are the same?

- ☐ Yes
☐ No

17.11. You are at point A and you want to know the distance to point B. However, point B is inaccessible (it is on the other side of a river, for example), so you cannot measure AB directly. Instead you proceed as follows. From A measure along a straight line at right angles to AB a length AC and bisect it at D. From C draw CE at right angles to CA on the side of it remote from B, and let E be the point on it which is in a straight line with B and D. The sought distance AB is equal to the measurable distance:

- ☐ DC

- ☐ DE
- ☐ CE
- ☐ AC

17.12. (cont.) The Euclidean propositions directly involved in setting up and justifying this procedure are:

- ☐ 1
- ☐ 2
- ☐ 4
- ☐ 5
- ☐ 7
- ☐ 8
- ☐ 9
- ☐ 10
- ☐ 11
- ☐ 13
- ☐ 14
- ☐ 15
- ☐ 16
- ☐ 22
- ☐ 23
- ☐ 26

17.13. Simple bookshelves (such as the IKEA Ivar) consist of two vertical and some horizontal planks. A problem is that they could tip askew, so that when we look at the bookshelf from the front we see, instead of a rectangle with vertical sides, a parallelogram with its longer sides inclined a few degrees with respect to the floor. Two ways of addressing this are often implemented in such bookshelves. The X method is to nail one or two long metal fortifiers diagonally across the back of the shelf. This forces the distance between diagonally opposite points to be fixed. The L method is to nail L-shaped metal fortifiers where the shelves meet the sides. This forces the angle between them to be fixed at 90 degrees. Match each method with a proposition from Euclid that can be used to explain it.

- ☐ X method
- ☐ L method
- ☐ Neither
- ☐ Proposition 4 (SAS), Proposition 8 (SSS), Proposition 26 (ASA)

§ 18. Early modern reflections on mathematical method

- 18.1. ☞ What did Descartes see as the key aspects of geometrical reasoning that were to be generalised to other domains of thought?
- 18.2. ☞ What is the relation between God and geometry, according to Descartes?
- 18.3. ☞ What is the relation between geometry and practical construction, according to Descartes?
- 18.4. ☞ Descartes is famous for introducing algebraic methods in geometry. What did he see as the purpose of this?

RENÉ DESCARTES, *A Discourse on the Method*, 1637, translated by Ian Maclean, Oxford University Press, 2008.

“I was most keen on mathematics, because of its certainty and the incontrovertibility of its proofs; but I did not yet see its true use. Believing as I did that its only application was to the mechanical arts, I was astonished that nothing more exalted had been built on such sure and solid foundations.” (9 = AT 7)

“The long chains of reasonings, every one simple and easy, which geometers habitually employ to reach their most difficult proofs had given me cause to suppose that all those things which fall within the domain of human understanding follow on from each other in the same way, and that as long as one stops oneself taking anything to be true that is not true and sticks to the right order so as to deduce one thing from another, there can be nothing so remote that one cannot eventually reach it, nor so hidden that one cannot discover it. And I had little difficulty in determining those with which it was necessary to begin, for I already knew that I had to begin with the simplest and the easiest to understand; and considering that of all those who had up to now sought truth in the sphere of human knowledge, only mathematicians have been able to discover any proofs, that is, any certain and incontrovertible arguments, I did not doubt that I should begin as they had done.” (17-18 = AT 19)

Descartes’s general method:

“The first [principle of my method] was never to accept anything as true that I did not incontrovertibly know to be so; that is to say, carefully to avoid both prejudice and premature conclusions; and to include nothing in my judgements other than that which presented itself to my mind so clearly and distinctly, that I would have no occasion to doubt it.” (17 = AT 18)

“The second was to divide all the difficulties under examination into as many parts as possible, and as many as was required to solve them in the best way.” (17 = AT 18)

“The third was to conduct my thoughts in a given order, beginning with the simplest and most easily understood objects, and gradually ascending, as it were step by step, to the knowl-

edge of the most complex; and positing an order even on those which do not have a natural order or precedence.” (17 = AT 18)

“The last was to undertake such complete enumerations and such general surveys that I would be sure to have left nothing out.” (17 = AT 19)

RENÉ DESCARTES, *Principles of Philosophy*, 1644, translated by R. P. Miller, Springer, 1982.

Descartes’s philosophical method is modelled on the method of Euclid’s *Elements*, as is clear from Descartes’s preface:

“One must begin by searching for ... first causes, that is, for Principles [which] must be so clear and so evident that the human mind cannot doubt of their truth when it attentively considers them ... And then, one must attempt to deduce from these Principles the knowledge of the things which depend upon them, in such a way that there is nothing in the whole sequence of deductions which one makes from them which is not very manifest.” (xvii-xviii)

But Descartes is not content with merely adopting the Euclidean method—he also justifies it. He does this by showing that it survives even the most critical examination possible, namely that announced in the first sentence of the text: “whoever is searching for truth must, once in his life, doubt all things” (I.1).

The Euclidean method is the only philosophical method to survive this critical abyss, by the following chain of reasoning.

First we prove our own existence. “We can indeed easily suppose that there is no God, no heaven, no material bodies; and yet even that we ourselves have no hands, or feet, in short, no body; yet we do not on that account suppose that we, who are thinking such things, are nothing: for it is contradictory for us to believe that that which thinks, at the very time when it is thinking, does not exist. And, accordingly, this knowledge, *I think, therefore I am*, is the first and most certain to be acquired by and present itself to anyone who is philosophizing in correct order.” (I.7)

“The knowledge of remaining things depend on a knowledge of God,” because the next things the mind feels certain of are basic mathematical facts, but it cannot trust these judgments unless it knows that its creator is not deceitful. Thus “the mind ... discovers [in itself] certain common notions, and forms various proofs from these; and as long as it is concentrating on these proofs it is entirely convinced that they are true. Thus, for example, the mind has in itself the ideas of numbers and figures, and also has among its common notions, *that if equals are added to equals, the results will be equal*, and other similar ones; from which it is easily proved that the three angles of a triangle are equal to two right angles, etc.” But the mind “does not yet know whether it was perhaps created of such a nature that it errs even in those things which appear most evident to it.” Therefore “the mind sees that it rightly doubts such things, and cannot have any certain knowledge until it has come to

know the author of its origin.” (I.13)

The existence of God is established to Descartes’s satisfaction by several dubious arguments, most notably the following. “Just as, for example, the mind is entirely convinced that a triangle has three angles which are equal to two right angles, because it perceives that the fact that its three angles equal two right angles is necessarily contained in the idea of a triangle; so, solely because it perceives that necessary and eternal existence is contained in the idea of a supremely perfect being, the mind must clearly conclude that a supremely perfect being exists.” (I.14) And all the more since it is “very well known from [our] natural enlightenment” “that that which is more perfect is not produced by an efficient and total cause which is less perfect; and moreover that there cannot be in us the idea or image of anything, of which there does not exist somewhere (either in us or outside us), some Original, which truly contains all its perfections. And because we in no way find in ourselves those supreme perfections of which we have the idea; from that fact alone we rightly conclude that they exist, or certainly once existed, in something different from us; that is, in God.” (I.18)

“It follows from this that all the things which we clearly perceive are true, and that the doubts previously listed are removed” (I.30), since “God is not the cause of errors,” owing to his perfection, seeing as “the will to deceive certainly never proceeds from anything other than malice, or fear, or weakness; and, consequently, cannot occur in God.” (I.29) “Thus, Mathematical truths must no longer be mistrusted by us, since they are most manifest.” (I.30)

In the same way we can be sure that material objects exist, since otherwise “it would be impossible to devise any reason for not thinking Him a deceiver” (II.1). But the argument forces upon us the restriction “that the nature of body does not consist in weight, hardness, color, or other similar properties; but in extension alone” (II.4), since a body can easily be conceived to be deprived of its secondary properties (cf. also II.11), but not its extension.

Physics, therefore, must be based on a theory of extended matter and nothing else. Two key characteristics of Cartesian physics follow quite naturally from this starting point, and are indeed introduced almost immediately: relativity of space (II.13-14) and contact mechanics (II.36-52).

The first is a quite unavoidable corollary of Descartes’s starting point, since his perspective does not admit the possibility of space as a concept separate from body. Thus he is compelled to argue that “the names ‘place’ or ‘space’ do not signify a thing different from the body which is said to be in the place; but only designate its size, shape and situation among other bodies” (II.13). “So when we say that a thing is in a certain place, we understand only that it is in a certain situation in relation to other things” (II.14).

A second rather straightforward consequence of Descartes’s starting point is that contact mechanics is the fundamental phenomena in terms of which all other physics must be construed. And indeed Descartes offers a detailed account of contact mechanics almost at once, in II.36-52.

RENÉ DESCARTES, *Rules for the direction of the mind*, Liberal Arts Press, 1961.

Mathematics is the model for all knowledge. "In seeking the correct path to truth we should be concerned with nothing about which we cannot have a certainty equal to that of the demonstrations of arithmetic and geometry." (II, 367) "And although I speak a good deal here of figures and numbers ... nevertheless anyone who pays close attention to my meaning will easily observe that I am not thinking at all of common mathematics, but I am setting forth a certain new discipline ... broad enough to bring out the truths of any subject whatsoever." (IV, 375) The power of mathematics stems from "certain basic roots of truth implanted in the human mind by nature, which we extinguish in ourselves daily by reading and hearing many varied errors" (IV, 377).

Proofs should be intuited as wholes. Deductions "may sometimes be accomplished through such a long chain of inferences that when we have arrived at the conclusions we do not easily remember the whole procedure which led us to them ... Because of this, I have learned to consider each of these steps by a certain continuous process of the imagination ... Thus I go from first to last so quickly that by entrusting almost no parts of the process to the memory, I seem to grasp the whole series at once." (VII, 388-389) "In this way our knowledge is made much more certain and the capacity of our minds is increased as much as possible." (XI, 408)

To achieve this end "we must make use of every assistance of the intellect, the imagination, the senses, and the memory" (XII, 411). "By the aid of each faculty ... human efforts can serve to repair the deficiencies of the mind." (XII, 417). For example, "if the intellect proposes to examine something which can be related to the body it should produce in the imagination the most distinct idea of it possible; and in order to do this more readily, the object which this idea represents should be exhibited to the external senses." (XII, 417-418). "We are to do nothing from this point on without the aid of the imagination." (XIV, 444)

Algebra and analytic geometry is intended to be precisely such an aid to the intuition. For having recorded the steps of a proof in algebraic terms, "we can run through all of them in a very rapid movement of thought and grasp as many as possible at the same time" (XVI, 456). Intuiting the whole in this way is important to us "who are seeking evident and distinct knowledge of things; but not the arithmeticians, who are satisfied if they have discovered the number sought even though they have not noticed how it depends upon the given facts, although this latter is the only point in which science truly lies" (XVI, 459).

Another benefit of algebra in this regard. By algebra "we translate what we understand to be affirmed about magnitudes in general into that particular magnitude that we can most easily and distinctly picture in our imagination" (XIV, 442); specifically, "a magnitude should never be regarded in the imagination otherwise than as a line or a surface, even though it may

be called a 'cube' or a 'biquadratic'" (XVI, 457).

A quip on why we should denote the answers we seek by a letter such as x . "It frequently happens that individuals are so eager to investigate problems that they apply their capricious intelligence to finding a solution before they have determined by what signs they will recognize the object of their search, if they should stumble upon it by accident; these persons are no less foolish that would be a boy, sent somewhere by his master, who was so eager to obey that he started to run without waiting for instructions, and without knowing where he was ordered to go." (XIII, 435)

18.5. ☞ What aspects of mathematics can be construed as showing that its proofs merely demonstrate propositions logically without explaining why they are true?

18.6. ☞ How can mathematics be defended against the charge that it is inferior to other sciences because its proofs are not explanatory and causal?

PAOLO MANCOSU, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, Oxford University Press, 1996.

"The classical texts of Euclid, Archimedes, Apollonius, Pappus, ... constituted an extremely stable body of results whose absolute certainty had seldom been put into doubt. In the scholastic Aristotelian tradition a long list of commentators argued for this certainty by remarking that mathematics conforms to the specifications for a perfect science set down by Aristotle in his Posterior Analytics. However, during the middle of the sixteenth century several objections were raised against such justifications of the certainty of mathematics. The debate that ensued from such criticisms is known as the Quaestio de Certitudine Mathematicarum. The main issues raised by this debate were (a) Does mathematics fit the definition of Aristotelian science or does it fall short of it? This problem led in turn to a careful analysis of mathematical demonstrations. And (b) If the certainty of mathematics cannot be argued by appealing to its logical structure on what other grounds can we justify it?" (10)

"My opinion is that the mathematical disciplines are not proper sciences. ... To have science is to acquire knowledge of a thing through the cause on account of which the thing is. ... However, demonstration (I speak of the most perfect kind of demonstration) must depend upon those things which are 'per se' and proper to that which is demonstrated; indeed, those things which are accidental and in common are excluded from perfect demonstrations. But the mathematician neither considers the essence of quantity, nor treats of its affections as they flow from such essence, nor declares them by the proper causes on account of which they are in quantity, nor makes his demonstrations from proper and 'per se' but from common and accidental predicates. Thus mathematical doctrine is not

properly science.” (Pereyra, 1562; 13)

“The geometer proves [*Elements* I.32] that the triangle has three angles equal to two right ones on account of the fact that the external angle which results from extending the side of that triangle is equal to two angles of the same triangle which are opposed to it. Who does not see that this middle is not the cause of the property which is demonstrated? ... [The external angle] is related in an altogether accidental way to [the angle sum of the triangle]. Indeed, whether the side is produced and the external angle is formed or not, or rather even if we imagine that the production of the one side and the bringing about of the external angle is impossible, nonetheless that property will belong to the triangle; but, what else is the definition of an accident than what may belong or not belong to the thing without its corruption?” (Pereyra, 1562; 15)

“Piccolomini [1547] had given several arguments aimed at showing that mathematical propositions could not be proven by causal proofs. One of the arguments appealed to the authority of Proclus, who had remarked that in geometry there are propositions of equal value and dignity that can be proved from each other. [E.g., *Elements*] 1.35 and 1.36, which are the converse of each other. In Piccolomini’s opinion this argues for the noncausality of such theorems. Were they causal, Piccolomini said, the same proposition would be the cause of itself. But this is a contradiction since nothing can be its own cause. The problem that Piccolomini is addressing here is the following. In mathematics one can often prove that A implies B and that B implies A—that A and B are equivalent. However, if we read ‘implication’ as being more than just logical consequence, and we demand the antecedent acts as the ‘cause’ of the conclusion then we run into trouble in trying to make sense of those theorems in which antecedents and consequences can be reversed, since in the intuitive reading of the relation of causality if A ‘causes’ B then B cannot ‘cause’ A. For, were the latter allowed, one could infer that A ‘causes’ A. But this goes against the intuitive reading of causality since, as Piccolomini remarks, nothing can be its own cause.” (25)

VIKTOR BLÄSJÖ, *Transcendental curves in the Leibnizian calculus*, Elsevier, 2017.

Leibniz: “[Geometry] does demonstrate from causes. For it demonstrates figures from motion; from the motion of a point a line arises, from the motion of a line a surface, from the motion of a surface a body. ... Thus the constructions of figures are motions, and the properties of figures, being demonstrated from their constructions, therefore come from motion, and hence, *a priori*, from a cause.” (45)

ARTHUR SCHOPENHAUER, *Die Welt als Wille und Vorstellung*, 1819. Quoted from *The World as Will and Idea*, translated by R.B. Haldane & J. Kemp, 1909.

“If now with our conviction that perception is the primary source of all evidence, and that only direct or indirect connection with it is absolute truth; and further, that the shortest way to this is always the surest, as every interposition of concepts means exposure to many deceptions; if, I say, we now turn with this conviction to mathematics, as it was established as a science by Euclid, and has remained as a whole to our own day, we cannot help regarding the method it adopts, as strange and indeed perverted. We ask that every logical proof shall be traced back to an origin in perception; but mathematics, on the contrary, is at great pains deliberately to throw away the evidence of perception which is peculiar to it, and always at hand, that it may substitute for it a logical demonstration. This must seem to us like the action of a man who cuts off his legs in order to go on crutches, or like that of the prince in the *Triumph der Empfindsamkeit* who flees from the beautiful reality of nature, to delight in a stage scene that imitates it. ...

Instead of ... giving a thorough insight into the nature of the triangle, [Euclid] sets up certain disconnected arbitrarily chosen propositions concerning the triangle, and gives a logical ground of knowledge of them, through a laborious logical demonstration, based upon the principle of contradiction. Instead of an exhaustive knowledge of these space-relations we therefore receive merely certain results of them, imparted to us at pleasure, and in fact we are very much in the position of a man to whom the different effects of an ingenious machine are shown, but from whom its inner connection and construction are withheld. We are compelled by the principle of contradiction to admit that what Euclid demonstrates is true, but we do not comprehend why it is so. We have therefore almost the same uncomfortable feeling that we experience after a juggling trick, and, in fact, most of Euclid’s demonstrations are remarkably like such feats. The truth almost always enters by the back door, for it manifests itself per accidens through some contingent circumstance. Often a *reductio ad absurdum* shuts all the doors one after another, until only one is left through which we are therefore compelled to enter. Often, as in the proposition of Pythagoras, lines are drawn, we don’t know why, and it afterwards appears that they were traps which close unexpectedly and take prisoner the assent of the astonished learner. ... The proposition of Pythagoras teaches us a *qualitas occulta* of the right-angled triangle. ...

This specially empirical and unscientific knowledge is like that of the doctor who knows both the disease and the cure for it, but does not know the connection between them. But all this is the necessary consequence if we capriciously reject the special kind of proof and evidence of one species of knowledge, and forcibly introduce in its stead a kind which is quite foreign to its nature. However, in other respects the manner in which this has been accomplished by Euclid deserves all the praise which has been bestowed on him through so many centuries, and which has been carried so far that his method of treating mathematics has been set up as the pattern of all scientific exposition. Men tried indeed to model all the sciences after it, but later they gave up the attempt without quite knowing why. Yet in our eyes this method of Euclid in mathematics can appear

only as a very brilliant piece of perversity.” (Volume I, Book I, §15)

§ 19. Elements 27–32, 46: parallels

19.1. Euclid’s theory of parallels (Propositions 27–31) studies whether two given lines are parallel or not by cutting them with a third line (which may be called the “test line”), and then looking at the angles produced. One can look at either the “alternate angles” (the pair of angles involved in Proposition 27; also called “Z-angles” because of their configuration) or the “internal angles” (the pair of angles involved in Postulate 5).

- ☐ Euclid uses alternate angles/Proposition 16 ...
- ☐ Euclid uses internal angles/Postulate 5 ...
- ☐ ... when he wants to prove that given lines are parallel.
- ☐ ... to derive properties any parallel lines must have.

19.2. Which of the following does Euclid establish without relying on the parallel postulate in any way?

- ☐ Parallel lines exist.
- ☐ There is precisely one unique parallel to a given line through a given point.
- ☐ Alternate angles (Z-angles) are equal.
- ☐ Vertical angles (opposite pair in an X configuration) are equal.
- ☐ Angle sum of triangle is two right angles.
- ☐ Proposition 30.
- ☐ None of the above.

19.3. Proposition 31 uses nothing later than Proposition 27, so Euclid could have placed it earlier. Do the intermediate propositions add some illumination?

- ☐ Yes, they imply that the parallel constructed in Proposition 31 is unique.
- ☐ Yes, they imply that the parallel constructed in Proposition 31 is not unique.
- ☐ Yes, they justify the implicit assumption in Proposition 31 that a parallel line exists.
- ☐ None of the above.

19.4. Parallel lines can also be characterised as lines that always have the same distance between them. But the notion of not crossing is arguably more fundamental. As Postulate 2 says, lines are fundamentally things that can be extended indefinitely. Asking whether they will cross seems quite natural already from this standpoint alone, whereas talking about the distance between lines requires us to involve a lot more secondary concepts to specify what it even means.

The geometry of other surfaces also problematise the equivalence of parallelism and equidistance in interesting ways. For example, it can happen that the curve equidistant to a straight line is not a straight line. In which of the following geometries does this occur?

- ☐ sphere
- ☐ 450 degree cone with stretched-string definition of straight line
- ☐ 450 degree cone with half-turn-symmetry definition of straight line

If we wanted to base the theory of parallels on the notion of equidistance we would have to prove that these kinds of situations do not occur.

19.5. Think about how Proposition 32 is related to Proposition 16. Why didn’t Euclid prove the more powerful 32 first and then derive 16 very easily from there?

- ☐ Euclid’s proof of Proposition 32 is logically dependent on Proposition 16, so 16 was a necessary stepping-stone along the way; without 16 we could not have reached 32.
- ☐ Although Euclid’s proof of Proposition 32 is logically independent of Proposition 16, Euclid’s way of proving Proposition 16 shows that it is independent of the parallel postulate, which would not have been shown if it was derived as a corollary of Proposition 32.

19.6. Why did Euclid wait so long (until Proposition 29) to use the parallel postulate? Does this tell us something about the status of the postulate?

19.7. The parallel postulate is considerably more convoluted than the other postulates. But it transpires from the *Elements* itself that Euclid could have used a simpler, equivalent statement in place of it, such as: given any line and any point not on this line, there is no more than one parallel to the line through that point. Why did Euclid choose his formulation of the postulate?

19.8. Proposition 47. An odd phrase occurs: “either ... or”. This phrasing in effect amounts to an implicit application of Proposition ☐.

§ 20. Rationalism versus empiricism

20.1. ☞ What aspects of classical geometry would you highlight if you wanted to justify the rationalistic interpretation? The empiricist one?

20.2. ☞ Newton and Leibniz disagreed on what it means to treat gravity scientifically. In what way does their disagreement parallel their views on the nature of geometry?

“For two millennia the method embodied in Euclid’s *Elements* was the gold standard of exact reasoning. By the time of the Renaissance and the scientific revolution it had also passed the test of time with flying colours: while it seemed that all other teachings invariably crumbled in the face of expanding knowledge and experience, the Euclidean edifice not only stood without a scratch but also proved an indispensable foundation for the most exciting new advances in the understanding of the world. The obvious message was not lost on reflective minds: If you seek certain and eternal truth then you better do whatever it was that Euclid did.

But what was it about the Euclidean method that made it so uniquely successful, and how could it be generalised beyond its traditional scope? Today the phrase ‘axiomatic-deductive method’ is often used to try to capture its essence, and indeed it is based on a small set of axioms, and indeed it proceeds meticulously through short, stringently verified deductive steps. But 17th century eyes saw something more in Euclid, something to which subsequent generations have grown increasingly blind. To them the ideal of the Euclidean method represented not a specialised, formal way of studying geometry, but a model of reasoning in general and our only reliable window toward an understanding of the nature of knowledge.

There were in fact two competing interpretations of the Euclidean method in the 17th century. They are summarised and contrasted in [Table 1]. As we see, these interpretations generalise the Euclidean method not only to an expanded view of geometry but also to physics and even philosophy in general. Descartes’s famous phrase *cogito ergo sum* (‘I think therefore I am’) encapsulates his view: one starts in complete ignorance and nothingness and can only build up one’s knowledge from the most immediate and undeniable principles. Newton’s view, by contrast, is summed up in his statement: ‘As in mathematics, so in natural philosophy, the investigation of difficult things by the method of analysis ought ever to precede the method of composition.’ That is to say, instead of the Cartesian method of ‘composing’ all knowledge from intuitive starting principles, Newton advocates its opposite: analysis, i.e., starting with all the things one wants to understand and then trying to reduce them to simple principles. Euclid’s *Elements* and Newton’s *Principia* both start with a few simple axioms and deduce increasingly more complex results from them, but this, according to Newton, is not to be seen as mirroring the process of acquiring knowledge. This ‘method of composition,’ or synthesis, is but a mode of presentation adopted after the fact, for the sake of consolidating and clarifying logically the insights gained through analysis.” (205–206)

“The defining characteristic of Cartesian physics is its insistence on explaining everything in terms of contact mechanics. Leibniz agreed completely: ‘A body is never moved naturally except by another body that touches and pushes it ... Any other

kind of operation on bodies in either miraculous or imaginary.’ Whence his famous conflict with Newton on the nature of gravity. Leibniz condemns very fiercely the notion of gravity as a primitive cause: ‘I maintain that the attraction of bodies, properly called, is a miraculous thing, since it cannot be explained by the nature of bodies.’ ... [Newton on the other hand was] largely content to simply stipulate gravity. ‘With the cause of gravity [I] meddle not’, says Newton, since ‘I have so little fancy to things of this nature.’” (58)

Another way of putting it is this. It would be very convenient to postulate:

- In classical geometry, solutions of classical construction problems.
- In early modern geometry, formulas as definitions of curves.
- In physics (as shown by Newton), action at a distance and absolute space.

But, from an operationalist point of view, it would be epistemologically irresponsible to do so. Instead one must speak only of what is:

- Generated by very simple processes.

Because then the intuitive warrant of starting principles propagate through the entire theory.

Geometrical constructions by ruler, compass, and other basic tools have this property. As does the mechanistic program of reducing physics to the collisions of bodies.

Newtonian action at a distance, by contrast, violates this principle.

- Reducible to concrete operations grounded in physical experience.

Because operational statements are meaningful, knowable, testable; non-operational statements often not. Experience is consistent; speculative thought often not.

The operational reading of geometrical theorems has exactly these properties, as we have seen. In physics, the same logic leads to the insistence that spatial relations are always relative because that is the only empirical knowledge we can have of them: by measurements we can determine whether a body is moving with respect to another, but not whether it is “really” moving in some absolute sense. Newton, on the contrary, stipulated an absolute space as a universal frame of reference for his physics, so that the velocity and position of a body have a fixed meaning in and of itself, not only in relation to other bodies, which violates the principle and is not an operationally meaningful notion.

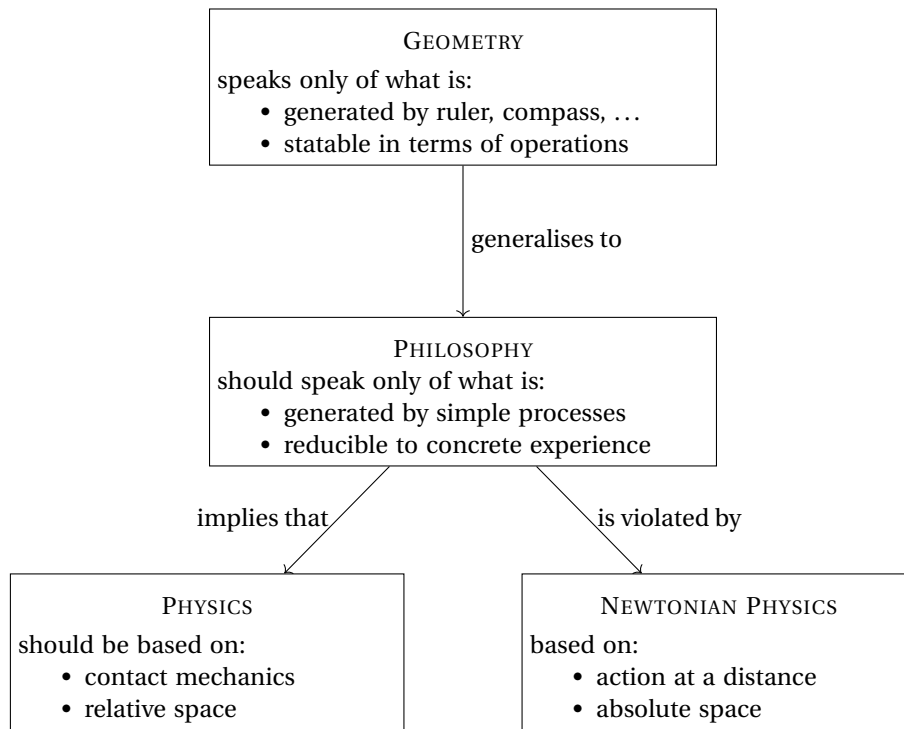


Figure 4: 17th-century operationalist view of scientific method.

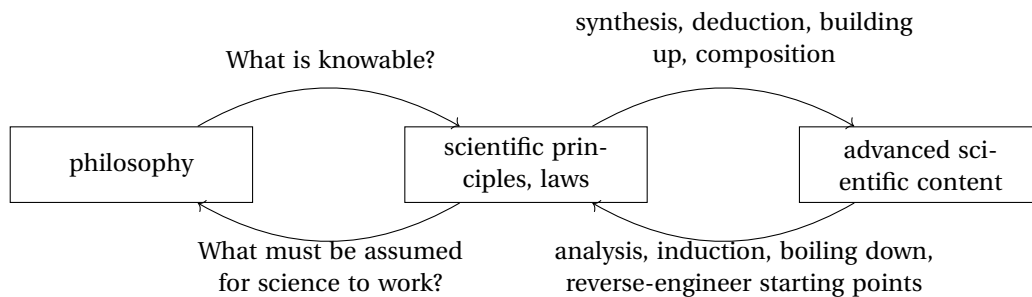


Figure 5: Opposite views on the relation between science and philosophy. Top: The view of Descartes, Leibniz, et al., according to which science must flow from philosophically justified starting principles. Bottom: The view of Newton, according to which philosophical and scientific principles are subordinated to science and retrofitted to agree with science after the fact.

	Descartes, Leibniz Continental rationalism	Newton British empiricism
The search for knowledge starts with ...	intuitively clear primitive notions	the rich diversity of phenomena
... and consists in ...	deducing the diversity of phenomena from them.	reducing them to a few simple principles.
Intrinsic justification of the axiomatic principles is ...	immediate by their intuitive nature	external to the matter at hand
... and is therefore ...	the crucial epistemological cornerstone of the entire enterprise.	of secondary importance at best.
In the case of physics, the axiomatic principles are ...	the laws of contact mechanics	Newton's three force laws and the law of gravity
... which are established by means of ...	their intuitively immediate nature.	induction from the phenomena.
In the case of geometry, the study of curves starts with ...	the primitive intuition of local motion	the diversity of curves conceived in any exact manner whatever
... and consists in ...	constructively building up a theory of all knowable curves on this basis.	investigating their properties in a systematic fashion.
Geometrical axioms are thus ...	the intuitively immediate principles that define and generate the entire subject.	the outcomes of the reductive study of curves, which it was found convenient and illuminating to take as assumptions when the time came to write a systematic account.
The certainty of geometrical reasoning ...	stems directly from the axioms' intuitive warrant and the constructive manner in which the rest is built up from them.	stems not from the axioms as such, but from the general method and exactitude of geometrical reasoning.

Table 1: Overview of the two competing interpretations of the Euclidean method in the 17th century. (From Blåsjö, 206.)

“The description of right lines and circles, upon which Geometry is founded, belongs to Mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn. For it requires that the learner should first be taught to describe these accurately, before he enters upon Geometry; then it shews how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from Mechanics; and by geometry the use of them, when so solved, is shewn. And it is the glory of Geometry that from those few principles, fetched from without, it is able to produce so many things. Therefore Geometry is founded in mechanical practice.”

RENÉ DESCARTES, *Principia philosophiae*, 1644. Quoted from *Principles of Philosophy*, Synthese Historical Library, Reidel, 1982.

“The names ‘place’ or ‘space’ do not signify a thing different from the body which is said to be in the place; but only designate its size, shape and situation among other bodies.” (II.13)
 “So when we say that a thing is in a certain place, we understand only that it is in a certain situation in relation to other things.” (II.14)

20.3. ☞ What led Descartes and Leibniz to insist on a relativistic notion of space? And Newton on an absolutist one?

GOTTFRIED WILHELM LEIBNIZ, 1689, quoted from R. Ariew and D. Garber (eds.), *Philosophical Essays*, Hackett, 1989.

20.4. ☞ What are the strengths and weaknesses of each view?

“Motion ... is nothing but a change in the positions of bodies with respect to one another, and so, motion is not something absolute, but consists in a relation.” (91)

20.5. ☞ How does this relate to the rationalist/empiricist divide?

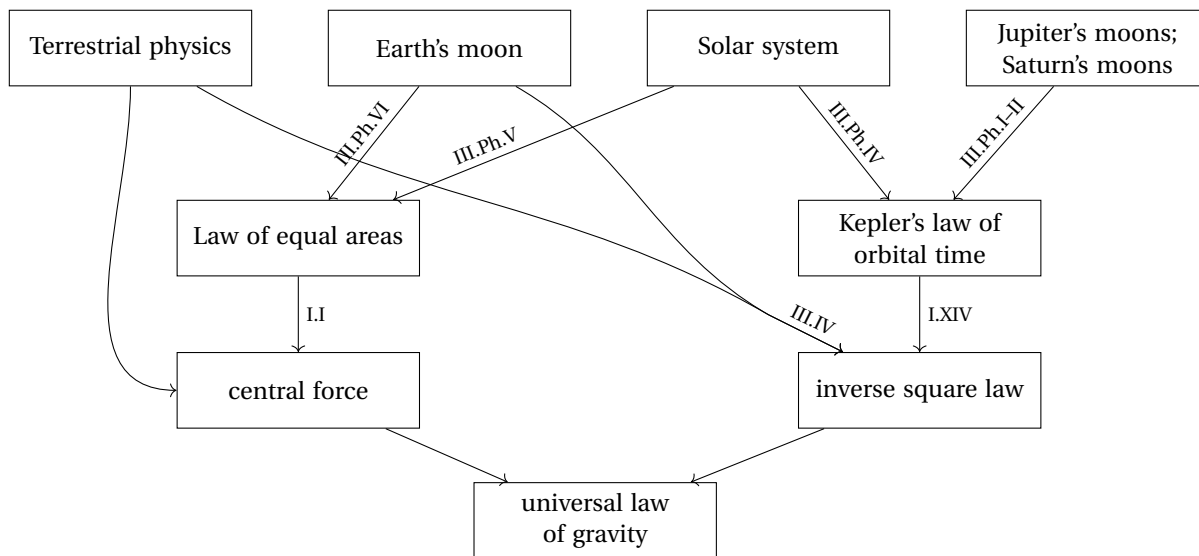


Figure 6: Flowchart illustrating Newton's method of reducing "phenomena" to universal scientific principles in the *Principia*.

ISAAC NEWTON, *Philosophiae Naturalis Principia Mathematica*, 1687, Book 1, Scholium to Definitions; trans. Andrew Motte..

I do not define time, space, place and motion, as being well known to all. Only I must observe, that the vulgar conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which, it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.

I. Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external. ... Relative, apparent, and common time, is some sensible and external measure of duration. ...

II. Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position [relative] to bodies; and which is vulgarly taken for immovable space. ... Because the parts of space cannot be seen, or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from the positions and distances of things from any body considered as immovable, we define all places; and then with respect to such places, we estimate all motions. ... And so, instead of absolute places and motions, we use relative ones; and that without any inconvenience in common affairs; but in philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them. For it may be that there is no body really at rest, to which the places and motions of others may be referred.

IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; or that part of its cavity which the body fills, and which therefore moves together with the ship: and relative rest is the continuance of the body in the same part of the ship, or of its cavity. But real, absolute rest, is the continuance of the body in the same part of that immovable space, in which the ship itself, its cavity, and all that it contains, is moved. Wherefore, if the earth is really at rest, the body, which relatively rests in the ship, will really and absolutely move with the same velocity which the ship has on the earth. But if the earth also moves, the true and absolute motion of the body will arise, partly from the true motion of the earth, in immovable space; partly from the relative motion of the ship on the earth; and if the body moves also relatively in the ship; its true motion will arise, partly from the true motion of the earth, in immovable space, and partly from the relative motions as well of the ship on the earth, as of the body in the ship; and from these relative motions will arise the relative motion of the body on the earth.

ERNST MACH, *The Science of Mechanics: A Critical and Historical Account of its Development*, Open Court Publishing, 1919, first edition 1883.

Absolute space and absolute motion ... are pure things of thought, pure mental constructs, that cannot be produced in experience. ... [They are] therefore neither a practical nor a scientific value; and no one is justified in saying that he knows aught about it. It is an idle metaphysical conception. ...

All our principles of mechanics are ... experimental knowledge concerning the relative positions and motions of bodies. ... No one is warranted in extending these principles beyond the

boundaries of experience. In fact, such an extension is meaningless, as no one possesses the requisite knowledge to make use of it. (II.VI)

MATTHIAS SCHEMMELE, *Historical Epistemology of Space: from Primate Cognition to Spacetime Physics*, Springer, 2016.

“Inertia ... served Newton as a proof of the independent existence of space. In the following reception of the concept of absolute space, it was exactly this feature which led to its broad acceptance. It allowed the concept of absolute space to serve as a foundation for a theory of mechanics, which was highly successful in integrating the growing body of knowledge on terrestrial and celestial motions. This emphasis on the mechanical argument is strikingly evident in Leonhard Euler’s *Reflections on Space and Time* [1748]. Euler makes no attempt to provide a metaphysical argument for the reality of space. Rather, he takes the empirical success of Newtonian mechanics as a proof that reality must pertain to space and time, since they serve as a basis for the concepts of absolute rest and uniform motion in a straight line, which are needed for the foundation of that science. He concludes that any metaphysical derivation that denies this reality can thereby be inferred to be flawed, without further analysis[:] ‘I do not want to enter the discussion of the objections that are made against the reality of space and place; since having demonstrated that this reality can no longer be drawn into doubt, it follows necessarily that all these objections must be poorly founded; even if we were not in a position to respond to them.’” (76–77)

ROBERT DISALLE, *Understanding Space-Time: The Philosophical Development of Physics from Newton to Einstein*, Cambridge University Press, 2006.

Empirical knowledge of spatial relations is always relative: by measurements we can determine whether a body is moving with respect to another, but not whether it is “really” moving in some absolute sense. Descartes and Leibniz insisted on this. Newton, on the contrary, stipulated an absolute space as a universal frame of reference for his physics, so that the velocity and position of a body have a fixed meaning in and of itself, not only in relation to other bodies.

“What right did Newton have to explain the observable relative motions by an appeal to these unobservable entities? What role can such metaphysical hypotheses play in empirical science? ... [In fact,] Newton’s theory of space and time was never a mere metaphysical hypothesis. Instead, it was his attempt to define the concepts presupposed by the new mechanical science.” (13) “[Newton] parted from his contemporaries ... in his belief that these novel explications [of space and time] came from science itself—that their authority rested, not on their conformity to epistemological and metaphysical principles ..., but on the role that they play in science. What must be the na-

ture of space and time, in order for the world to be as it appears to be, and to follow the natural laws that it appears to follow? This is Newton’s question.” (42)

“Newton’s theory was forced into confrontation with the most prominent general philosophical accounts of space and time, namely those of Descartes and Leibniz. But its rejoinder to them was only that those philosophical views could not be reconciled with their own views of physics.” (55) Indeed, there is an “apparent contradiction within Leibniz’s picture: ... How can the idea of force as a genuine metaphysical quantity be reconciled with the relativity of motion? Leibniz himself occasionally juxtaposed these notions, apparently unaware of the conflict that others saw quite clearly. ... For example, he wrote, ‘... each [body in a group of interacting bodies] does truly have a certain degree of motion, or, if you wish, of force, in spite of the equivalence of these hypotheses about their motion’. ... As Euler put it, ‘Therefore I am least afraid of those philosophers who reduce everything to relations, since they themselves attribute so much to motion that they regard moving force as something substantial’.” (61)

Similarly, the law of inertia—that a body under no external influence continues to move in the same direction with the same speed—seems almost impossible to formulate coherently in a relativistic framework. As Euler put it: “For if space and place were nothing but the relation among co-existing bodies, what would be the same direction? ... Identity of direction, which is an essential circumstance in the general principles of motion, is ... not to be explicated by the relation ... of co-existing bodies.” (37)

“For the [relativists], then, there were only three legitimate ways to resist Newton’s ... argument. One would be to acknowledge it, but to insist on its limitations: ... [it falls short of proving that absolute space is unavoidable, leaving the possibility open that] a weaker structure than absolute space [might suffice]. ... A second way is to argue that, since the laws of motion do presuppose something like the Newtonian conception of space, time, and motion, they ought to be replaced by laws that don’t presuppose any such thing. ... The third way is ... to maintain that no matter what spatio-temporal concepts might be required by physics, the authority to pronounce upon the true nature of space, time, and motion ultimately belongs to metaphysics.” (52)

Options 1 and 2 were far from viable in the 17th century, though eventually, in the late 19th century, they proved the way forward. “But we can point to a lasting philosophical accomplishment of Newton’s work. He showed that the philosophical understanding of space and time has to start, not from general philosophical principles, but from a critical analysis of what we presuppose in our observation and reasoning about the physics of motion. The eventual overthrow of Newton’s theory was made possible by the further pursuit, in a different theoretical and empirical context, of the same kind of analysis.” (53)

§ 21. Abstract axiomatics

For thousands of years, Euclid's parallel postulate made mathematicians a bit uncomfortable. Many felt that it ought to be a theorem rather than a postulate. Various strategies for proving it were devised. One strategy was to try to prove it by contradiction, that is, to assume it to be false and try to show that this led to absurd consequences. Indeed, it was found that negating the parallel postulate had various strange consequences that felt “repugnant to the nature of the straight line” (as Saccheri put it in *Euclides Vindicatus*, 1733). For example, if the parallel postulate is false then squares do not exist. It is difficult to imagine a world in which squares are impossible. But weird is not the same as self-contradictory. Despite their best efforts, mathematicians could not find a clear-cut proof that negating the parallel postulate led to directly contradictory conclusions.

This line of inquiry pushed mathematicians toward an ever-more abstract, logico-formalistic conception of geometry. We can see why. Reasoning about things contrary to our imagination requires us to use abstract logic rather than intuition and visualisation. And basing our conclusions on gut feelings about what is “repugnant to the nature of the straight line” and the like is contrary to the geometrical ideal of deducing everything from explicitly formulated assumptions. The modern trend toward formalisation and abstraction culminated in the work of Hilbert, a world-leading mathematician and author of the seminal *Grundlagen der Geometrie* (1899). He summed up the spirit of this development like this: “one must be able to say at all times—instead of points, straight lines, and planes—tables, chairs, and beer mugs.”⁴ That is to say, the terms of geometry must be taken to have meaning only through formally specified axioms, without any connotations or assumptions being imported from psychological associations or intuitive conceptions of the entities in question.

In this section we shall practice thinking in the Hilbert way. This will help us when we take on non-Euclidean geometry later.

Investigating the status of the parallel postulate in the manner outlined above leads to the question: what can we prove *without* using the parallel postulate? More generally, a good way to understand the logical relations between axioms is to study what follows from a few of them at a time. Carrying this perspective to its extreme, we may ask ourselves: what are the minimum assumptions needed for something to be called geometry at all? Discarding the parallel postulate impoverishes us, but it still leaves us with lines and points and circle and triangles and so on, so it's still geometry. How much further could we go? How many more of our assumptions could we throw out? Here is one possible bare minimum set of assumptions for geometry:

Axiom 1. If L is a line, then there exists at least two points belonging to L .

Axiom 2. If L is a line, then there exists at least one

point which is not on L .

Axiom 3. There exists at least one line.

Axiom 4. If A and B are different points, then there exists at least one line which contains both A and B .

From a logical point of view, “line” and “point” are undefined terms. Any formal system must involve undefined terms because everything must be defined in terms of something, and those things in terms of something else, and so on. If we did not allow undefined terms this process would never end.

Axioms 1–4 are obviously true if we think of the usual Euclidean meaning of line and point. But it is also true for the geometry of a sphere, or a cylinder, or a cone, and so on. Each of these are a “model” of the axiom system—that is, they are ways of giving meaning to the undefined terms that make the axioms true. In fact, we can imagine much more exotic models.

21.1. Which of the following are models of the minimal geometry axioms?

- ☐ The points are: all grains of sand in Africa. The lines are: all possible sets of such points.
- ☐ The points are: all the points inside an ordinary circle. The lines are: all the cords of this circle (that is, all the line segments connecting two points on the circumference).
- ☐ The points are: all stations in the Stockholm metro system. The lines are: all the lines of the network (T10, T11, T13, T14, T17, T18, T19), each consisting of all the stations on that route.



- ☐ The points are: artichoke, Beatrice, cold. The lines are: the line consisting of artichoke and cold, and the line consisting of Beatrice and cold.
- ☐ The points are: the three vertices of a triangle. The lines are: the three sides of that triangle, each consisting of its two endpoints.

21.2. Is the statement “each point belongs to some line” always true in any model of Axioms 1–4? In any model of three of the axioms? Select all sets of axioms sufficient to make the statement true.

⁴David Hilbert, *Gesammelte Abhandlungen*, III.403.

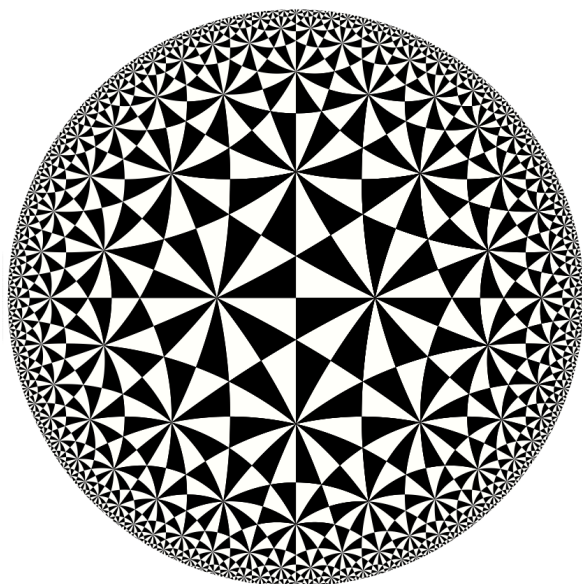
- ☐ 1, 2, 3
- ☐ 1, 2, 4
- ☐ 1, 3, 4
- ☐ 2, 3, 4
- ☐ 1, 2, 3, 4
- ☐ None of the above.

21.3. Select all that are true for any model of Axioms 1-4.

- ☐ Each point belongs to at least two different lines.
- ☐ Each point belongs to at least three different lines.
- ☐ For any point there exists a line that does not contain that point.
- ☐ There exists four points such that not all four are on the same line.

§ 22. Hyperbolic geometry

This is a representation of the so-called hyperbolic plane:



Intrinsically, these tiles are all of equal size. Thus the space represented in this picture is actually infinite since the tiles become smaller and smaller as you approach the boundary. A bug living in this world could never walk to the boundary; there would always be more tiles to go. The lines in this world are arcs of circles perpendicular to the boundary (including the diameters of the disc, which are part of circles with infinite radius, so to speak). Hyperbolic angles are the same as the Euclidean angles in the picture.

A useful tool for visualisations and interactive constructions in hyperbolic geometry is: www.cs.unm.edu/~joel/NonEuclid/

22.1. Argue that lines in this sense do indeed seem to represent the shortest (hyperbolic) distances between any two points.

22.2. Which postulates hold in hyperbolic geometry?

- ☐ Postulate 1
- ☐ Postulate 1' (Euclid's Postulate 1 with the added requirement that the line is unique)
- ☐ Postulate 2
- ☐ Postulate 3
- ☐ Postulate 4
- ☐ Postulate 5

22.3. What does this mean for the feasibility of proving the parallel postulate from the other axioms?

22.4. Which geometries are "locally Euclidean" (i.e., practically indistinguishable from Euclidean geometry if one "zooms in far enough," i.e., restricts oneself to a small region)?

- ☐ 450 degree cone
- ☐ Spherical geometry
- ☐ Hyperbolic geometry

22.5. For a given line L and a given point P in hyperbolic geometry, which are possible?

- ☐ There are no parallels to L through P .
- ☐ There is precisely one parallel to L through P .
- ☐ There is a finite number of parallels to L through P .
- ☐ There are infinitely many parallels to L through P .

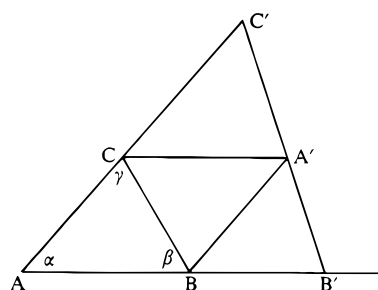
22.6. (a) Name some other ways in which hyperbolic lines differ from plane and spherical lines.

(b) Name some ways in which hyperbolic and plane lines differ from spherical lines.

22.7. Argue that the angle sum of a hyperbolic triangle is less than 180° .

22.8. Do we live in a Euclidean or hyperbolic world? How can you tell?

Legendre tried to prove that the angle sum of a triangle cannot be less than 180° using only the first four axioms of Euclid. His proof goes as follows:



Let ABC be a triangle with angles α, β, γ such that $\alpha + \beta + \gamma < 180^\circ$. Call $180^\circ - (\alpha + \beta + \gamma) = \delta$ the defect of the triangle. Locate A' symmetrically situated to A with respect to BC (this can be done by rotating ABC through 180° around the midpoint of BC) and

extend AB and AC. Draw through A' a line meeting AB at B' and AC at C'—not necessarily a line 'parallel' to BC which would beg the question. Join up A'B and A'C. By symmetry the defect of triangle A'BC is also δ . Since the defect of the angle sum of the large triangle AB'C' is the sum of the defects in the angle sum of the four triangles separately, we have in AB'C' a triangle with defect $\geq 2\delta$. Continuing in this manner we obtain triangles with defects $\delta, 2\delta, 4\delta, 8\delta$ and so on. Eventually, then, we will have a triangle whose defect is greater than 180° , which is absurd since this would mean that $\alpha + \beta + \gamma$ is a negative number.

We have thus showed that if there is a triangle with angle sum $< 180^\circ$, then there are also triangles with angle sum less than any given number, including even 0° . Therefore there can be no triangle with angle sum $< 180^\circ$.

22.9. Why does Legendre's proof regarding the angle sum of a triangle not work in the hyperbolic plane?

- ☐ The placement of A'BC and its congruence with ABC assumes properties of rotations that do not hold in hyperbolic geometry.
- ☐ When constructing C'A'B', it is assumed that for any pair of intersecting lines, and any point between them, one can draw a line through that point crossing both the given lines. But this assumption does not hold in the hyperbolic plane.
- ☐ It is assumed that the angular defect of the big triangle is the sum of the angular defects of its subtriangles. This is based on assuming that a "straight angle" is 180 degrees. But this assumption does not hold in the hyperbolic plane.
- ☐ Repeating the process produces triangles with greater and greater defect, but the defect does not go as low as 0; rather it approaches an upper bound, as the angle sum of the triangle approaches the original angle alpha.

Lagrange tried to prove the uniqueness of parallels from the principle that if there are two parallels to a given line through a given point, then the reflection of one of these parallels in the other should also be a parallel of the given line. This principle seems reasonable since there is no evident reason why one side of a line should be "privileged" over the other; either it's parallel or it's not; one line can't be "more parallel" than another. By reflecting again and again we can create new parallels further and further away from the ones we started with, which clearly produces a contradiction.

22.10. Why does Lagrange's proof regarding uniqueness of parallels not work in the hyperbolic plane?

- ☐ His principle is false, because when a parallel is reflected in a parallel, the result is not necessarily another parallel.

- ☐ His principle still holds but the conclusion he draws from it does not follow, because repeated reflection does not necessarily generate new, "more spread out" parallels.
- ☐ His principle holds, as does the generation of ever more parallels through repeated reflection, but the assumption that this leads to a contradiction is false.

§ 23. Space and perception

23.1. ☞ What is the relation between geometry and empirical data regarding spatial relations, according to Kant?

23.2. ☞ What does Kant's view accomplish?

23.3. ☞ How does Kant's view relate to the issues of rationalism versus empiricism and absolute versus relative space?

23.4. ☞ Is geometrical knowledge in some sense specifically human or tied to human nature? Compare Kant with Plato, Kepler, Descartes in this regard.

IMMANUEL KANT, *Inaugural Dissertation*, 1770.

"Space is not something objective and real ...; instead, it is subjective and ideal, and originates from the mind's nature ... as a scheme, as it were, for coordinating everything sensed externally." (Ak 2: 403)

IMMANUEL KANT, *The Critique of Pure Reason*, second edition, 1787, translated by J. M. D. Meiklejohn.

"A question which requires close investigation [is:] whether there exists a knowledge altogether independent of experience, and even of all sensuous impressions? Knowledge of this kind is called *a priori*, in contradistinction to empirical knowledge, which has its sources *a posteriori*, that is, in experience." (Introduction, §1)

"Certain of our cognitions rise completely above the sphere of all possible experience, and by means of conceptions, to which there exists in the whole extent of experience no corresponding object, seem to extend the range of our judgements beyond its bounds. ... In this transcendental or supersensible sphere ... experience affords us neither instruction nor guidance." (Introduction, §3)

"Mathematical science affords us a brilliant example, how far, independently of all experience, we may carry our *a priori* knowledge. ... [It] thus leads us to form flattering expectations with regard to others, though these may be of quite a different nature. (Introduction, §3)

"Deceived by such a proof of the power of reason, we can perceive no limits to the extension of our knowledge. The light

dove cleaving in free flight the thin air, whose resistance it feels, might imagine that her movements would be far more free and rapid in airless space. Just in the same way did Plato, abandoning the world of sense because of the narrow limits it sets to the understanding, venture upon the wings of ideas beyond it, into the void space of pure intellect. He did not reflect that he made no real progress by all his efforts; for he met with no resistance which might serve him for a support, as it were, whereon to rest, and on which he might apply his powers, in order to let the intellect acquire momentum for its progress. It is, indeed, the common fate of human reason in speculation, to finish the imposing edifice of thought as rapidly as possible, and then for the first time to begin to examine whether the foundation is a solid one or no. Arrived at this point, all sorts of excuses are sought after, in order to console us for its want of stability, or rather, indeed, to enable us to dispense altogether with so late and dangerous an investigation. But what frees us during the process of building from all apprehension or suspicion, and flatters us into the belief of its solidity, is this. A great part, perhaps the greatest part, of the business of our reason consists in the analysis of the conceptions which we already possess of objects. By this means we gain a multitude of cognitions, which although really nothing more than elucidations or explanations of that which (though in a confused manner) was already thought in our conceptions, are, at least in respect of their form, prized as new introspections; whilst, so far as regards their matter or content, we have really made no addition to our conceptions, but only disinvolved them. But as this process does furnish a real *priori* knowledge, which has a sure progress and useful results, reason, deceived by this, slips in, without being itself aware of it, assertions of a quite different kind; in which, to given conceptions it adds others, *a priori* indeed, but entirely foreign to them, without our knowing how it arrives at these, and, indeed, without such a question ever suggesting itself. I shall therefore at once proceed to examine the difference between these two modes of knowledge. (Introduction, §3)

"Of the difference between analytical and synthetical judgments. ... Analytical judgments ... add in the predicate nothing to the conception of the subject, but only analyse it into its constituent conceptions, which were thought already in the subject, although in a confused manner; [synthetical judgments] add to our conceptions of the subject a predicate which was not contained in it, and which no analysis could ever have discovered therein. For example, when I say, 'all bodies are extended', this is an analytical judgment. For I need not go beyond the conception of body in order to find extension connected with it, but merely analyse the conception, that is, become conscious of the manifold properties which I think in that conception, in order to discover this predicate in it: it is therefore an analytical judgment. On the other hand, when I say, 'all bodies are heavy', the predicate is something totally different from that which I think in the mere conception of a body. By the addition of such a predicate, therefore, it becomes a synthetical judgment." (Introduction, §4)

"Mathematical judgments are always synthetical. ... 'A straight

line between two points is the shortest', is a synthetical proposition. For my conception of straight, contains no notion of quantity, but is merely qualitative. The conception of the shortest is therefore wholly an addition, and by no analysis can it be extracted from our conception of a straight line." (Introduction, §5)

"Geometry is a science which determines the properties of space synthetically, and yet *a priori*." (Transcendental Aesthetic, §3)

"It has hitherto been assumed that our cognition must conform to the objects; but all attempts to ascertain anything about these objects *a priori*, by means of conceptions, and thus to extend the range of our knowledge, have been rendered abortive by this assumption. Let us then make the experiment whether we may not be more successful in metaphysics, if we assume that the objects must conform to our cognition. This appears, at all events, to accord better with the possibility of our gaining the end we have in view, that is to say, of arriving at the cognition of objects *a priori*, of determining something with respect to these objects, before they are given to us. ... If the intuition must conform to the nature of the objects, I do not see how we can know anything of them *a priori*. If, on the other hand, the object conforms to the nature of our faculty of intuition, I can then easily conceive the possibility of such an *a priori* knowledge." (preface to 2nd ed.)

"It is therefore from the human point of view only that we can speak of space, extended objects, etc. If we depart from the subjective condition, under which alone we can obtain external intuition, or, in other words, by means of which we are affected by objects, the representation of space has no meaning whatsoever. This predicate is only applicable to things in so far as they appear to us, that is, are objects of sensibility." (Transcendental Aesthetic, §4)

ROBERT DISALLE, *Understanding Space-Time: The Philosophical Development of Physics from Newton to Einstein*, Cambridge University Press, 2006.

"[Newton's approach] cannot be viewed as a complete philosophical account of space and time, ... because it treats space and time solely from the perspective of classical mechanics—that is as concepts implicitly presupposed by the classical mechanical understanding of causality and force. A philosophically thorough treatment of the problem would embrace, not only the implicit metaphysics of physics, but the general epistemological problem of space and time and the ways in which physics, and human knowledge generally, have some access to them." (55) In Kant's own words, his goal was to "provide, not engineers, as Euler had in mind, but geometers themselves with a convincing ground ... for claiming the actuality of their absolute space." (62)

"[Newton had showed that] the concept of absolute motion ... was implicitly assumed in ... physics, and was in a sense a 'condition of possibility' of ... reasoning about the motions of the

solar system and their physical causes. ... [But this is a limited form of argument since it shows only that] the concept is necessary, relative to a certain well-established practice of scientific reasoning about a certain kind of phenomenon. Kant, instead, argues that the Newtonian concept, and the laws of motion, [are necessary in much more general sense:] They are the *only* basis on which the concept of causality can be applied to the universe at large. They are the only basis, indeed, on which the phenomena of the heavens can be grasped as something more than mere appearances—as the appearance of genuine physical objects that stand in objective geometrical and causal relations.” (71)

-
- 23.5. ☞ What is the relation between geometry and sensory perception?
- 23.6. ☞ How, and under what conditions, can a mind without geometrical preconceptions be led to impose a geometrical structure on sensory perceptions?
- 23.7. ☞ Does the experience of blind people becoming sighted support this account?
- 23.8. ☞ What is Helmholtz's critique of Kant? How might a Kantian reply to it?

HENRI POINCARÉ, *La Valeur de la Science*, 1905, quoted from *The Foundations of Science*, translated by George Bruce Halsted, The Science Press, 1913.

“The movements of our body [plays] the preponderant role ... in the genesis of the notion of space. For a being completely immovable there would be neither space nor geometry; in vain would exterior objects be displaced about him, the variations which these displacements would make in his impressions would not be attributed by this being to changes of position, but to simple changes of state; this being would have no means of distinguishing these two sorts of changes, and this distinction, fundamental for us, would have no meaning for him.

... We are led to distinguish the changes produced by our own motions and we easily discriminate them for two reasons: because they are voluntary; because they are accompanied by muscular sensations. So we naturally divide the changes that our impressions may undergo into two categories ...: the internal changes, which are voluntary and accompanied by muscular sensations; the external changes, having the opposite characteristics.

We then observe that among the external changes are some which can be corrected, thanks to an internal change which brings everything back to the primitive state; others can not be corrected in this way (it is thus that, when an exterior object is displaced, we may then by changing our own position replace ourselves as regards this object in the same relative position as before, so as to reestablish the original aggregate

of impressions; if this object was not displaced, but changed its state, that is impossible). Thence comes a new distinction among external changes: those which may be so corrected we call changes of position; and the others, changes of state.

Think, for example, of a sphere with one hemisphere blue and the other red; it first presents to us the blue hemisphere, then it so revolves as to present the red hemisphere. Now think of a spherical vase containing a blue liquid which becomes red in consequence of a chemical reaction. In both cases the sensation of red has replaced that of blue; our senses have experienced the same impressions which have succeeded each other in the same order, and yet these two changes are regarded by us as very different; the first is a displacement, the second a change of state. Why? Because in the first case it is sufficient for me to go around the sphere to place myself opposite the blue hemisphere and reestablish the original blue sensation. ...

Another example: An object is displaced before my eye; its image was first formed at the center of the retina; then it is formed at the border. ... Why ... am I led to decide that these two sensations, qualitatively different, represent the same image, which has been displaced? It is because I can follow the object with the eye and by a displacement of the eye, voluntary and accompanied by muscular sensations, bring back the image to the center of the retina and reestablish the primitive sensation.

I suppose that the image of a red object has gone from the center to the border of the retina, then that the image of a blue object goes in its turn from the center to the border of the retina; I shall decide that these two objects have undergone the same displacement. Why? Because in both cases I shall have been able to reestablish the primitive sensation, and that to do it I shall have had to execute the same movement of the eye, and I shall know that my eye has executed the same movement because I shall have felt the same muscular sensations.

If I could not move my eye, should I have any reason to suppose that the sensation of red at the center of the retina is to the sensation of red at the border of the retina as that of blue at the center is to that of blue at the border? I should only have four sensations qualitatively different, and if I were asked if they are connected by the proportion I have just stated, the question would seem to me ridiculous, just as if I were asked if there is an analogous proportion between an auditory sensation, a tactile sensation and an olfactory sensation.” (I.III.5)

RICHARD FEYNMAN, *The Feynman Lectures on Physics*, Addison-Wesley, 1963.

“When we look at an object, there is an obvious thing we might call the ‘apparent width’, and another we might call the ‘depth’. But the two ideas, width and depth, are not fundamental properties of the object, because if we step aside and look at the same thing from a different angle, we get a different width and a different depth, and we may develop some formulas for computing the new ones from the old ones and the angles involved. ... If it were impossible ever to move, and we always saw a given

object from the same position, then this whole business would be irrelevant—[width and depth] would appear to have quite different qualities, because one appears as a subtended optical angle and the other involves some focusing of the eyes ...; they would seem to be very different things and would never get mixed up. It is because we can walk around that we realize that depth and width are, somehow or other, just two different aspects of the same thing.

[In Einstein's theory of special relativity] also we have a mixture—of positions and the time. ... In the space measurements of one man there is mixed in a little bit of the time, as seen by the other. Our analogy permits us to generate this idea: The 'reality' of an object that we are looking at is somehow greater (speaking crudely and intuitively) than its 'width' and its 'depth' because they depend upon how we look at it; when we move to a new position, our brain immediately recalculates the width and the depth. But our brain does not immediately recalculate coordinates and time when we move at high speed, because we have had no effective experience of going nearly as fast as light to appreciate the fact that time and space are also of the same nature. It is as though we were always stuck in the position of having to look at just the width of something, not being able to move our heads appreciably one way or the other." (I.17-1)

WILL. CHESSELDEN, An account of some observations made by a young gentleman, who was born blind, or lost his sight so early, that he had no remembrance of ever having seen, and was couch'd between 13 and 14 Years of age, *Philosophical Transactions*, 35(402), 1728, 447–450.

"When he first saw, he was so far from making any Judgment about Distances, that he thought all Objects whatever touch'd his Eyes, (as he express'd it) as what he felt, did his Skin. ... He knew not the Shape of any Thing, nor any one Thing from another, however different In Shape, or Magnitude; but upon being told what Things were, whose Form he before knew from feeling, he would carefully observe, that he might know them again; but having too many Objects to learn at once, he forgot many of them. ... One Particular only (tho' it may appear trifling) I will relate; Having often forgot which was the Cat, and which the Dog, he was asham'd to ask; but catching the Cat (which he knew by feeling) he was observ'd to look at her stedfastly, and then setting her down, said, So Puss! I shall know you another Time. He was very much surpriz'd, that those Things which he had lik'd best, did not appear most agreeable to his Eyes, expecting those Persons would appear most beautiful that he lov'd most, and such Things to be most agreeable to his Sight that were so to his Taste. We thought he soon knew what Pictures represented, which were shew'd to him, but we found afterwards we were mistaken; for about two Months after he was couch'd, he discovered at once, they represented solid Bodies; when to that Time he consider'd them only as Party-colour'd Planes, or Surfaces diversified with Variety of Paint; but even then he was no less surpriz'd, expect-

ing the Pictures would feel like the Things they represented, and was amaz'd when he found those Parts, which by their Light and Shadow appear'd now round and uneven, felt only flat like the rest; and ask'd which was the lying Sense, Feeling, or Seeing? ... Being shewn his Father's Picture in a Locket at his Mother's Watch, and told what it was, he acknowledged a Likeness, but was vastly surpriz'd; asking, how it could be, that a large Face could be express'd in so little Room, saying, It should have seem'd as impossible to him, as to put a Bushel of any thing into a Pint."

JEAN JACQUES ROUSSEAU, *Emile, or On Education*, 1762. Quoted from the 2013 Dover Publications edition, p. 36.

"It is only by our own movements that we gain the idea of space. The child has not this idea, so he stretches out his hand to seize the object within his reach or that which is a hundred paces from him. You take this as a sign of tyranny, an attempt to bid the thing draw near, or to bid you bring it. Nothing of the kind, it is merely that the object first seen in his brain, then before his eyes, now seems close to his arms, and he has no idea of space beyond his reach."

HERMANN VON HELMHOLTZ, The Facts in Perception, 1878, quoted from *Epistemological Writings*, Reidel, 1977.

"Kant ... developed the doctrine of forms of intuiting and thinking given prior to all experience ... into which forms any content we may represent must necessarily be absorbed. ... [Similarly,] Locke had already established established a claim for the share which our corporeal and mental makeup has in the manner in which things appear to us. In this direction, investigations into the physiology of the senses ... have now brought the fullest confirmation." (118–119)

"Sensations belonging to different senses ... exclude any transition from one to the other, any relationship of greater or lesser similarity. One cannot at all ask whether e.g. sweet is more similar to blue or to red. On the other hand, ... between different sensations of the same sense ... transition and comparison are possible. We can ... e.g. declare yellow to be more similar to orange than to blue. What physiological investigations now show is that [this] deeply incisive difference does not depend, in any manner whatsoever, upon the kind of external impression whereby the sensation is excited, but is determined alone and exclusively by the the sensory nerve upon which the impression impinges." (119) "The same aether vibrations as are felt by the eye as light, are felt by the skin as heat. The same air vibrations as are felt by the skin as a quivering motion, are felt by the ear as a note." (120)

"So physiology too acknowledges the qualities of sensation to be a mere form of intuition. But Kant went further. ... Kant considers spatial specifications too as belonging as little to the world of the actual—or to 'the thing in itself'—as the colours

which we see [as] attributes of bodies in themselves, but [which are actually] introduced by our eye into them.” (123) “Kant assumed not only that the general form of spatial intuition is transcendently given, but that it also contains in advance, and prior to any experience, certain narrower specifications as expressed in the axioms of geometry. ... [Such as:] Between two points only *one* shortest line is possible. ... Through any point only one line parallel to a given straight line is possible. ... Kant used the alleged fact that these propositions appeared to us as *necessarily* correct, and that we could never at all even represent to ourselves a deviating behaviour of space, directly as a proof that they had to be given prior to all experience.” (128)

“When Kant asserted that spatial relationships contradicting the axioms of Euclid could never in any way be represented, he was influenced by the contemporary states of the development of mathematics and the physiology of the senses, just as he was thus influenced in his whole conception of intuition in general as a simple psychic process, incapable of further resolution.” (129)

“The concepts of [non-Euclidean] spatial structures ... can readily be developed only by means of calculative analytic geometry. ... With this the intuitability of such spaces, in the sense [of a coherent mathematical system], has been shown. ... This is however in disagreement with the older concept of intuition, which only acknowledges something to be given through intuition if its representation enters consciousness at once with the sense impression, and without deliberation and effort. Our attempts to represent [non-Euclidean] spaces indeed do not have the ease, rapidity and striking self-evidence with which we for example perceive the form of a room which we enter for the first time, together with the arrangement and forms of the objects contained in it, the materials of which these consist, and much else besides. Thus if this kind of self-evidence were an originally given an necessary peculiarity of all intuition, we could not up to now assert the intuitability of such spaces.” (130–131)

“Yet ... assurance and rapidity for the occurrence of specific representations with specific impressions can also be acquired—even when no such connexion is given by nature. One of the most striking examples of this kind is our understanding of our mother tongue. Its words are arbitrarily or accidentally chosen signs—every different language has different ones. Understanding of it is not inherited, since for a German child who was brought up amongst Frenchmen and has never heard German spoken, German is a foreign language. The child becomes acquainted with the meaning of the words and sentences only through examples of their use. In this process one cannot even make understandable to the child—until it understands the language—that the sounds it hears are supposed to be signs having a sense. Lastly, on growing up it understands these words and sentences without deliberation and effort, without knowing when, where, and through what examples it learnt them, and it grasps the finest variations of their sense—often ones where attempts at logical definition only limp clumsily behind.” (131)

“Thus as concerns ... the issue of the origin of the axioms of

geometry: the fact that the representation of [non-Euclidean] spatial relationships is not easy when experience is lacking, cannot be claimed as a ground against their intuitability. ... Kant’s proof for the transcendental nature of the axioms of geometry is thus inadequate. On the other hand, investigation of the empirical facts shows that the axioms of geometry, taken in the only sense in which one is allowed to apply them to the actual world, can be empirically tested and demonstrated, or even—if the case should arise—refuted.” (132)

“The assumption that there is an acquaintance with the axioms which comes from transcendental intuition is: (1) an unproved hypothesis; (2) an unnecessary hypothesis, since it does not pretend to explain anything in what in fact is our world of representations which could not also be explained without its help; (3) a wholly unusable hypothesis for explaining our acquaintance with the actual world, since the propositions laid down by it should only ever be applied to the relationships in the actual world after their objective validity has been experimentally tested and ascertained.” (162)

§ 24. Abstract versus physical geometry

- 24.1. ☞ What is the modern view of the relation between geometry and empirical data? What are its strengths and weaknesses compared to older conceptions?
- 24.2. ☞ What elements of older views such as that of Kant can be salvaged in light of non-Euclidean geometry?
- 24.3. ☞ Has the history of the epistemology of geometry been one of constant retreats? That is, has the perceived status and claim to truth of geometrical knowledge been shrinking with each century?

CARL FRIEDRICH GAUSS, Quoted from Jeremy Gray, Gauss and Non-Euclidean Geometry, in András Prékopa & Emil Molnár (eds.), *Non-Euclidean Geometries*, Springer, 2006, 63.

“The assumption that the angle sum is less than 180° leads to a geometry quite different from Euclid’s, logically coherent, and one that I am entirely satisfied with. ... The theorems are paradoxical but not self-contradictory or illogical.” (1824) “The necessity of our [Euclidean] geometry cannot be proved. ... Geometry must stand, not with arithmetic which is pure *a priori*, but with mechanics.” (1817)

NIKOLAI LOBACHEVSKY, *Pangeometry*, 1855, European Mathematical Society, 2010.

“Pangeometry [i.e., non-Euclidean geometry] ... proves that the assumption that the value of the sum of the three angles of any right rectilinear triangle is constant, an assumption which is explicitly or implicitly adopted in ordinary geometry, is not

a consequence of our notions of space. Only experience can confirm the truth of this assumption, for instance, by effectively measuring the sum of three angles of a rectilinear triangle. ... One must give preference to triangles whose edges are very large, since according to Pangeometry, the difference between two right angles and the three angles of a rectilinear triangle increases as the edges increase." (75) "The distances between the celestial bodies provide us with a means for observing the angles of triangles whose edges are very large." (76)

ALBERT EINSTEIN, *Geometry and Experience*, 1921. *Collected Papers*, Volume 7.

"How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things? In my opinion the answer to this question is, briefly, this: as far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. It seems to me that complete clarity as to this state of things became common property only through that trend in mathematics, which is known by the name of 'axiomatics'. ...

Let us for a moment consider from this point of view any axiom of geometry, for instance, the following: through two points in space there always passes one and only one straight line. How is this axiom to be interpreted in the older sense and in the more modern sense?

The older interpretation: everyone knows what a straight line is, and what a point is. Whether this knowledge springs from an ability of the human mind or from experience, from some cooperation of the two or from some other source, is not for the mathematician to decide. He leaves the question to the philosopher. Being based upon this knowledge, which precedes all mathematics, the axiom stated above is, like all other axioms, self-evident, that is, it is the expression of a part of this *a priori* knowledge.

The more modern interpretation: geometry treats of objects, which are denoted by the words straight line, point, etc. No knowledge or intuition of these objects is assumed but only the validity of the axioms, such as the one stated above, which are to be taken in a purely formal sense, i.e., as void of all content of intuition or experience. These axioms are free creations of the human mind. All other propositions of geometry are logical inferences from the axioms (which are to be taken in the nominalistic sense only). ... In axiomatic geometry the words 'point', 'straight line', etc., stand only for empty conceptual schemata. That which gives them content is not relevant to mathematics. ...

'Practical geometry' [arises if we] add the proposition: solid bodies are related, with respect to their possible dispositions, as are bodies in Euclidean geometry of three dimensions. Then the propositions of Euclid contain affirmations as to the be-

havior of practically-rigid bodies. ... All length-measurements in physics constitute practical geometry in this sense, so, too, do geodetic and astronomical length measurements, if one utilizes the empirical law that light is propagated in a straight line, and indeed in a straight line in the sense of practical geometry. ... I attach special importance to the view of geometry, which I have just set forth, because without it I should have been unable to formulate the theory of relativity. ... From the latest results of the theory of relativity it is probable that our three-dimensional space is ... approximately spherical, that is, that the laws of disposition of rigid bodies in it are not given by Euclidean geometry, but approximately by spherical geometry."

HENRI POINCARÉ, *La Science et l'Hypothèse*, 1902, quoted from *The Foundations of Science*, translated by George Bruce Halsted, The Science Press, 1913.

"The axioms of geometry ... are neither synthetic *a priori* judgments nor experimental facts. They are conventions; our choice among all possible conventions is guided by experimental facts; but it remains free and is limited only by the necessity of avoiding all contradiction. Thus it is that the postulates can remain rigorously true even though the experimental laws which have determined their adoption are only approximate."

"In other words, the axioms of geometry ... are merely disguised definitions. Then what are we to think of that question: Is the Euclidean geometry true? It has no meaning. As well ask whether the metric system is true and the old measures false; whether Cartesian coordinates are true and polar coordinates false. One geometry can not be more true than another; it can only be more convenient."

"The question has also been put in another way. If [hyperbolic] geometry is true, the parallax of a very distant star will be finite [which is equivalent to an angle sum of less than 180° for an astronomical triangle]; if [spherical geometry] is true, it will be negative. These are results which seem within the reach of experiment, and there have been hopes that astronomical observations might enable us to decide between the three geometries. But in astronomy 'straight line' means simply 'path of a ray of light'. If therefore negative parallaxes were found, ... two courses would be open to us; we might either renounce Euclidean geometry, or else modify the laws of optics and suppose that light does not travel rigorously in a straight line. It is needless to add that all the world would regard the latter solution as the more advantageous. The Euclidean geometry has, therefore, nothing to fear from fresh experiments." (81)

More generally, "Is the position tenable, that certain phenomena, possible in Euclidean space, would be impossible in non-Euclidean space, so that experience, in establishing these phenomena, would directly contradict the non-Euclidean hypothesis? For my part I think no such question can be put. To my mind it is precisely equivalent to the following, whose absurdity is patent to all eyes: are there lengths expressible in meters and centimeters, but which can not be measured in fathoms,

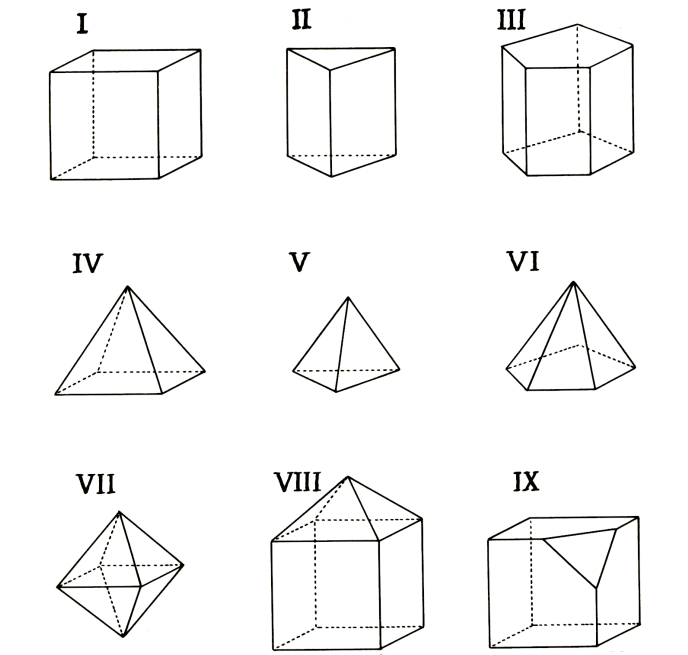
feet and inches, so that experience, in ascertaining the existence of these lengths, would directly contradict the hypothesis that there are fathoms divided into six feet?" (81–82)

ROBERT DiSALLE, *Understanding Space-Time: The Philosophical Development of Physics from Newton to Einstein*, Cambridge University Press, 2006.

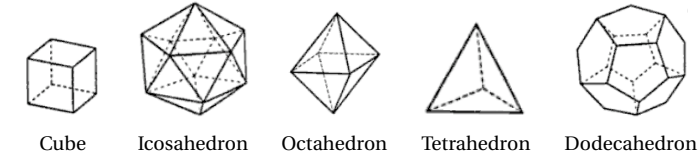
“Kant [was] right in a certain limited sense; the kind of a-priori principle that goes to constitute a spatio-temporal framework cannot be the same as an empirical principle, and cannot be justified by the usual sort of empirical or inductive argument—since empirical arguments ... must take such principles for granted.” (156) “In the history of modern physics, space and time have after all played something like the role attributed to them by Kant. Not as forms of intuition: this was only incidentally the case, in a context where the geometry of space and the intuitive means of knowing about space seemed inseparable from one another. ... But they have played the quasi-Kantian role of a framework that enables physics to constructively define its fundamental concepts.” (153)

§ 25. Polyhedra

Bodies composed of polygonal faces are called polyhedra.⁵



There are precisely five *regular* polyhedra; that is, polyhedra where the faces are regular polygons and the corners as all alike.



Is there a relationship between the number of faces F , vertices V , and edges E of a polyhedron? If we count these numbers for the above examples of polyhedra we find:

Polyhedron	F	V	E
I. cube	6	8	12
II. triangular prism	5	6	9
III. pentagonal prism	7	10	15
IV. square pyramid	5	5	8
V. triangular pyramid	4	4	6
VI. pentagonal pyramid	6	6	10
VII. octahedron	8	6	12
VIII. “tower”	9	9	16
IX. “truncated cube”	7	10	15

Looks quite complicated. But what if we sort the same data according to the last column instead?

Polyhedron	F	V	E
triangular pyramid	4	4	6
square pyramid	5	5	8
triangular prism	5	6	9
pentagonal pyramid	6	6	10
cube	6	8	12
octahedron	8	6	12
pentagonal prism	7	10	15
“truncated cube”	7	10	15
“tower”	9	9	16

And for the regular polyhedra:

Polyhedron	F	V	E
triangular pyramid	4	4	6
square pyramid	5	5	8
triangular prism	5	6	9
pentagonal pyramid	6	6	10
cube	6	8	12
octahedron	8	6	12
pentagonal prism	7	10	15
“truncated cube”	7	10	15
“tower”	9	9	16

25.1. Find a formula that expresses the relation between F , V , and E .

We could find a formula for these examples, but does the formula work for *any* polyhedron? One way of investigating this conjecture is to check whether the relationship expressed by the formula is preserved by various polyhedral manipulations.

25.2. Consider the following ways of manipulating a polyhedron. “Roofing,” as in going from I to VIII. “Truncating,” as in going from I to IX. “Pinching,” as in collapsing the top face of III to get VI. How are the quantities F , V , E

⁵This figure and subsequent tables are from G. Polya, *Induction and Analogy in Mathematics*, Princeton University Press, 1954.

changed under such an operation? (In the answer options below, n indicates the number of sides of the face to which the manipulation is applied, or, in the case of truncation, the number of edges meeting in the vertex to which the manipulation is applied.)

☐ Roofing, Truncating, Pinching

☐ $\begin{array}{rrr} -1 & -n+1 & -n \\ +1 & +n-1 & +n \\ +n-1 & +1 & +n \end{array}$

25.3. Suppose we started with a polyhedron for which the formula of problem 25.1 holds. In which cases do the formula also hold after the transformation has been applied?

- ☐ roofing
- ☐ truncating
- ☐ pinching
- ☐ none of these operations

25.4. A “soccer ball” polyhedron can be obtained from one of the polyhedra above by

- ☐ roofing
- ☐ truncating
- ☐ pinching
- ☐ none of these operations



Read the dialogue part (starting at page 8 and continuing to page 25) of Lakatos's paper “Proofs and Refutations” available at <https://www.jstor.org/stable/685347>. You do not need to pay attention to all the mathematical details. The interesting broader question is to what extent this kind of back-and-forth dynamic between proofs and refutations should be taken as typical of mathematics altogether.

25.5. One of the historical footnotes quite early on in the dialogue mentions crystals. What is the point being made?

- ☐ Crystals provided motivation for classifying polyhedra.
- ☐ Crystals provided motivation for focussing on V, E, E.
- ☐ Crystals suggested a counterexample to the formula.
- ☐ The formula can be used to understand how crystals form.

☐ The formula can be used to understand how crystals break.

25.6. Which of the characters in the dialogue think that trying to poke holes in arguments and looking for counterexamples for its own sake is a good way of doing mathematics?

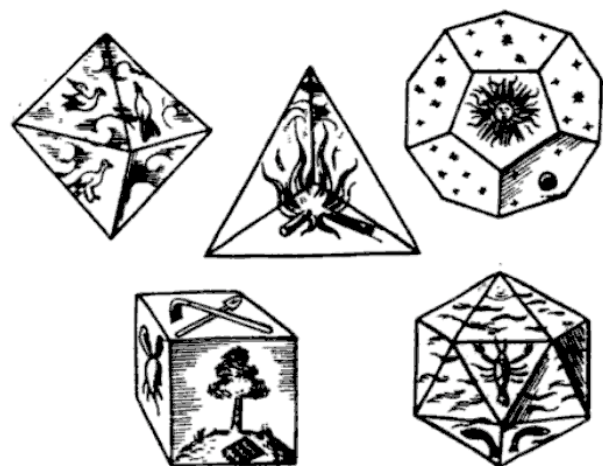
- ☐ Teacher
- ☐ Alfa
- ☐ Delta
- ☐ Gamma

25.7. Lakatos leaves us with the impression that mathematical theorems are:

- ☐ Conjectures that have not yet been refuted.
- ☐ Infallible by virtue of the precision of mathematical language.
- ☐ Socially constructed beliefs.

PLATO, *Timaeus*, translated by Donald J. Zeyl, Hackett, 2000.

Polyhedral theory of the elements. “Let us now assign to fire, earth, water, and air the [regular polyhedra]. To earth let us give the cube, because of the four kinds of bodies earth is the most immobile and the most pliable ... And of the solid figures that are left, we shall next assign the least mobile of them to water, to fire the most mobile, and to air the one in between” (55d-56a). The dodecahedron “still remained, and this one the god used for the whole universe” (55c).

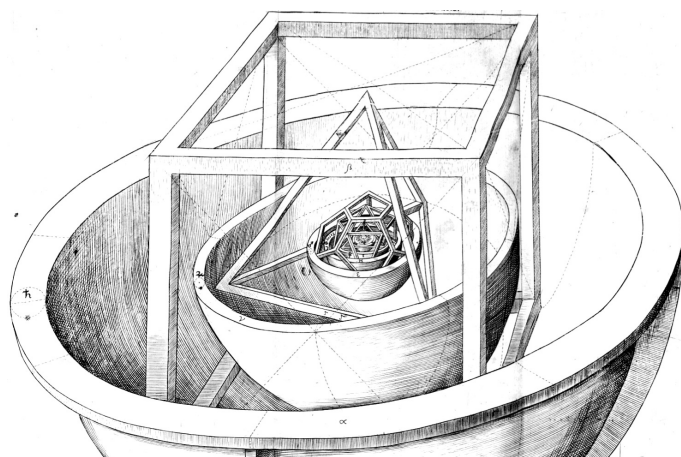


Applications of the polyhedral theory (“a moderate and sensible diversion,” 59d). Water=icosahedron has 20 equilateral triangles as its sides, while fire=tetrahedron has 4 and air=octahedron 8, so “when water is broken up into parts by fire or even by air, it could happen that the parts recombine to form one corpuscle of fire and two of air” (56d), i.e., steam is two parts air and one part fire. A second example may illustrate how the relative sizes of the polyhedra matter (61a). Fire is of

course the smallest, followed by air. Thus, for example, water can normally be dissolved by air (evaporation) by air octahedra slipping in between the water icosahedra. But since the fire tetrahedra are smaller they dissolve water much more efficiently. And if the water is sufficiently packed (ice) then air cannot dissolve it at all since only fire can get through the cracks.

JOHANNES KEPLER, *Mysterium Cosmographicum: The Secret of the Universe* [1596], ABaris Books, 1981.

“Greetings, friendly reader. The nature of the universe, God’s plan for creating it, God’s source for the numbers, ... the reason why there are six orbits, the spaces which fall between all the spheres ... —here Pythagoras reveals all this to you by five figures.” (49) Namely, there are five regular polyhedra, and the spheres of the planets are nested in such a way that the regular polyhedra fit precisely between them:



The ordering of the polyhedra in this arrangement is justified by a myriad arguments, of which the following is a representative sample:

“[The regular polyhedra] are classified into three primaries, the cube, tetrahedron and dodecahedron, and two secondaries, the octahedron and the icosahedron. For the correctness of this distinction, note the properties of each class. ... 2. Every one of the primaries has its particular type of face: the cube has the square, the pyramid the triangle, the dodecahedron the pentagon; the secondaries borrow the triangular face from the pyramid. ... 6. It is characteristic of the primaries to stand upright, of the secondaries to balance on a vertex. For if you roll the latter onto their base, or stand the former on a vertex, in either case the onlooker will avert his eyes at the awkwardness of the spectacle. ... Therefore, since there was an obvious distinction between the solids, nothing could be more appropriate than that our Earth, the pinnacle and pattern of the whole universe, and therefore the most important of the moving stars, should by its orbit differentiate between the two classes stated,

and should be allotted the position which we have attributed to it above.” (105)

§ 26. Cultural history of trigonometry

The following questions refer to these videos:

https://youtu.be/ozuEb_qLNys

<https://youtu.be/wnerBKI51bs>

<https://youtu.be/WFLWorXHE10>

<https://youtu.be/D4NFvqkhug0>

26.1. Match the cultural, geographical, or contextual aspects with their associated applications of trigonometry discussed in the videos.

- ☐ Religious ceremonies, Particular plants, Existence of mountains, Particular buildings, Expanding empire
- ☐ Eratosthenes, Chinese, al-Biruni motivation, al-Biruni choice of method, Snellius

26.2. Match the measurement methods from the videos with the assumptions or conditions needed for them.

- ☐ Horizon visibility, City-to-city visibility, Rays from sun parallel, Rays from sun not parallel, Eclipses
- ☐ Aristarchus, Eratosthenes, Chinese, al-Biruni, Snellius

§ 27. General discussion questions

- 27.1. ☞ Argue that geometry, as conceived in Euclid’s *Elements*, is distinct from the physical world; that it is purely logical, abstract, Platonic.
- 27.2. ☞ Argue that geometry, as conceived in Euclid’s *Elements*, concerns physical space and is anchored in concrete and hands-on operations with real-world tools like ruler and compass.
- 27.3. ☞ What are some examples of “assumption minimalism” in Euclid? That is, situations where it would have been natural to assume a bit more but Euclid would rather start from only the least possible assumption even if this means more work.
- 27.4. ☞ Should all of Euclid’s theorems be seen as essentially conditional statements—“if you accept the postulates, then logic compels you to also accept this theorem”—as opposed to truths in and of themselves?
- 27.5. ☞ If Greek civilisation had been eradicated in some war before it got going, would there have been “another Euclid” somewhere else?
- 27.6. ☞ If there’s intelligent life on another planet, do they know Euclidean geometry?