

## Frege's Justification of the Basic Laws

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### Abstract

Frege intended to prove that the truths of arithmetic can be grounded on our knowledge about logic. This raises a question as to how he intended to ground our knowledge about logic. Dummett, Hacker, and De Pierris maintain that he appealed to self-evidence at this juncture. Stroud argues that he left this problem unsolved. Conant argues that, for Frege, the negation of basic laws is impossible to grasp, which would have led him to self-defeating conclusions. I argue that, for Frege, basic laws are ideally impossible to reject.

Frege intended to show that our knowledge of arithmetic can be tracked back to our knowledge of the basic laws of logic. Now, imagine that Frege's basic laws could give a foundation of arithmetic as he envisioned; this would still leave open the question: how are we justified in believing the basic laws? In this paper, I discuss Frege's solution to this problem. My suggestion is that Frege (1983) maintained that the basic laws are justified because rejecting them would be ideally impossible.<sup>1</sup>

This solution, *prima facie*, seems both wrong and unsatisfactory. It seems wrong because, at least *prima facie*, people *can* reason illogically. Anyone who spent some time studying logic knows how easy it is to make logical mistakes.<sup>2</sup> It seems unsatisfactory because it is unclear how one could be justified in believing something just because one cannot think its negation. A subject could be unable, for whatever reason, to think that  $\pi$  approximates to 3.14, but that would hardly justify the subject in believing that  $\pi$  does not approximate to 3.14. This paper discusses how this solution can be made to look neither wrong nor unsatisfactory; or, better, I discuss why Frege thought that this was neither wrong nor unsatisfactory.<sup>3</sup> In the process, I argue against other interpretations. Let us briefly review these.

Wittgenstein contrasts his own approach to the justification of logic to what he takes to be Frege's and Russell's. In dictation, and in the *Tractatus*, he says that Frege and Russell appealed to self-evidence, thus holding a psychologistic epistemology of logic.

One of the great difficulties which is in logic was this—what is the criterion for a proposition being a proposition of logic.

By some it was taken to be self-evidence. This is what both Russell and Frege said. At certain points where this is a difficulty one tends to bring in psychology.

The word self-evident is taken from psychology.

It is what everybody in his right senses would say is true.

It is the highest degree of plausibility. But it is clear that what is self-evident [to somebody] may not be to another.<sup>4</sup> (WSM, 270)

Never mind whether this is a correct assessment of how Russell meant to justify basic logical principles. According to some interpreters, this is a faithful depiction of Frege's approach. For instance, De Pierris (1988) argues that Frege appealed to self-evidence.<sup>5</sup> Laws of logic "neither need nor admit of proof" (Frege 1884, 4). According to De Pierris: "Frege does not attempt to explain

what guarantees that our laws of logic are the laws of truth [he] treats [logic] as a piece of accepted knowledge" (1988, 310). Thus, all evidence we can give to support them is that they feel, as Russell puts it, "luminously evident" (Russell 1912, 64).<sup>6</sup>

One problem with this interpretation is that it suggests that Frege, the champion of anti-psychologism, placed a psychological element at the core of his system. Wittgenstein takes this to be a *reductio*, while other interpreters try to remove the psychological element in Frege's alleged appeal to self-evidence. For instance, Dummett (1980) argues that Frege's self-evidence is a non-psychological acquaintance with platonic entities, analogous to sense perception. And Stroud (2018) argues that Frege is not truly appealing to a technical epistemological notion of self-evidence, but rather he is leaving the secondary problem of pure reason unsolved:

"The question why and with what right we acknowledge a law of logic to be true," Frege says, "logic can answer only by reducing it to another law of logic." That leaves the 'ground' or justification of those laws themselves so far unexplained. It says nothing to suggest that there even could be such an explanation. (Stroud 2018, 210 quoting Frege 1893, xvii)

I call *the controversial passage* the paragraph in the *Grundgesetze* that Stroud is quoting here (Frege 1893, xvii). This is quoted by many interpreters, in full or in parts, when discussing Frege's views on the justification of logic (Dummett 1980, 510; Hacker 1986, 43; De Pierris 1988, 305-6; Conant 1992, 78). Now, Stroud's reading of the controversial passage might remove the psychological element from Frege's alleged appeal to self-evidence, but it does so at a great cost.

First, Stroud's interpretation breaks down Frege's analogy between logical laws and scientific laws. Macfarlane (2002) argued that Frege's laws of logic are *weakly normative*.<sup>7</sup> According to this interpretation, the laws of logic "are descriptive in content but have prescriptive consequences" (ibid. 37). Stroud seems to endorse this reading, as I do, but his interpretation of the controversial passage is in tension with it. Much of the strength of this weak normativism interpretation is that it makes laws of logic analogous to scientific laws, the only difference being their generality. But the analogy breaks down if we add that *unlike scientific laws*, the laws of logic cannot be justified.

Second, Stroud's view seems to leave a gap in Frege's theory. Our knowledge of arithmetic is justified by our knowledge of the laws of logic. But our knowledge of the laws of logic is unaccounted for. Laws of logic are either "a basic part of our repertoire or [...] just part of the way things are" (Stroud 2018, 217), but we seem to have no way of justifying this belief. Thus, in this view the ultimate ground of our arithmetical knowledge is somewhat unjustified. Stroud suggests that there is nothing problematic with this; it is merely a psychological fact that we tend to find this unsatisfactory. However, he gives no reason to think that Frege thought the same. Indeed, Stroud presents this as his own attitude. Accordingly, it is reasonable to have a sceptical attitude here, especially given what Frege writes concerning the epistemic status of axioms.

We cannot accept a thought as an axiom if we are in doubt about its truth. (Frege 1914, 205)

Similar remarks can be made about Dummett's (1980) reading. If the justification of laws of logic derives from direct acquaintance with platonic entities, then the laws of logic are different from scientific laws in the source of their justification. But Frege seems to think that the only difference should be that laws of logic are more general than any other law (Frege 1893, xv). Furthermore, Dummett's view also seems to leave a gap in Frege's theory, since the idea that we can be directly acquainted with platonic entities is somewhat unqualified. As De Pierris puts it: "the view does not seem to have more content than this metaphor or simile with the direct acquaintance with physical objects by means of sense perception" (1988, 297).

These difficulties give us reason, at least *prima facie*, to search an alternative reading. At this juncture, Conant suggests that, for Frege, the laws of logic are "constitutive of the possibility of

thought" (1992, 57). While Stroud maintains that no *unmistaken* thought can violate Frege's laws of logic, Conant maintains that no thought *at all* can violate them and this somehow justifies them.

This reading removes the gap in Frege's theory. There is an explanation as to how we can be justified to believe the laws of thought. And this explanation is somewhat analogous to the explanation we may give about natural laws. The laws of motion, for instance, are laws that no possible moving body can violate, and scientists study moving bodies to determine what laws of motion there are. Analogously, the laws of thought are laws that no possible thought can violate, and logicians study thoughts to determine what laws of thought there are. The only difference is that laws of thought are more general than laws of motion.

This reading, however, makes Frege's view self-defeating. As argued by Ricketts (1985), there is a tension between the idea that illogical thought is impossible and other aspects of Frege's theory. I show this here considering semantic compositionality, and the idea that the laws of logic have a sense. Frege, according to Conant, claims that violations of a law of logic are not thoughts at all. For instance, there is no thought corresponding to the formula  $\neg(P \rightarrow P)$ , this is just nonsense. Now, by semantic compositionality, the negation of nonsense is also nonsense. Therefore,  $\neg\neg(P \rightarrow P)$  is nonsense. Frege accepted double negation elimination, therefore this commits him to the idea that  $P \rightarrow P$  is nonsense. But this is an instance of basic law I!

It cannot be overestimated how great a cost this is for Frege. He maintains that the sense of a sentence is the way in which it displays its reference, and there are only two possible references of any sentence, namely the True and the False (Frege, 1982b). In this framework, the view that basic laws are nonsense entails that they fail to display the True. Therefore, the ground on which Frege meant to give a foundation of arithmetic crumbles under his feet.

I should emphasize that Conant is aware that his reading creates a tension in Frege's theory. He maintains that Frege did not realize that his views on the epistemology of logic were ultimately self-defeating.<sup>8</sup> Conant describes this as an intermediary step on the road to Wittgenstein's more perspicuous theory of arithmetic

The problem with Conant's reading is that it makes Frege defeat himself at a juncture where he was very unlikely to make a mistake. I argue that Conant's view entails that Frege confused the notions of *grasping* and *asserting*. Now, each of Frege's technical works begins by drawing the distinction between the judgement stroke and the horizontal or content stroke (Frege 1879 §2; 1893, §5).<sup>9</sup> This distinction corresponds to the grasping-asserting distinction. It seems very unlikely that Frege would confuse these very notions. Therefore, embracing Conant's reading should be considered a last resort. I argue that we can avoid making this gambit: Frege thought that the basic laws are ideally impossible to *reject*, not *grasp*.

The rest of this paper has the following structure. In the first section, I present the controversial passage from the *Grundgesetze* where Frege discusses the justification of the basic laws. In the second section, I discuss the self-evidence interpretation. In the third section, I discuss Conant's view and I put forward my own view. Finally, in the fourth section, I discuss my interpretation constructively. That is, I explain in what sense, for Frege it is impossible to reject the basic laws, and why this impossibility gives us a justification to believe them.

## I.

The central aim of Frege's logicism is finding the proof for every arithmetical truth and "following it up right back to the primitive truths" (Frege 1884, §3). These primitive truths, he wants to show, are the principles of logic as he conceived it. In the *Grundlagen*, Frege develops informal arguments to support this idea. For instance, he writes:

The basis of arithmetic lies deeper, it seems, than any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought? (Frege 1884, §14)

Frege's informal arguments in favour of logicism are meant as preliminary evidence. Only deriving arithmetical truths case by case, he says, could give a definitive proof that arithmetic is analytic (1884, §90).<sup>10</sup> In the *Grundgesetze*, Frege endeavours to complete his lifelong project.<sup>11</sup>

Frege's approach to logicism was doomed to fail because basic law V leads his system to a contradiction. But this paper is not about that part of the story. I focus on how Frege, when he still thought that his project would succeed, intended to justify the basic laws. If arithmetic stands in need of a justification, and this justification is to be sought in logic, then one might ask, does logic also stand in need for a justification? Could the foundation of arithmetic succeed if the ground on which it stood were not bedrock?

Frege seems compelled to address these questions. Indeed, he says "it is the nature of mathematics always to prefer proof, where proof is possible, to any confirmation by induction" (Frege 1884, §2). Given this, he should presumably try to prove the laws of thought. But this is rendered arduous by what Ricketts (1985) and Sheffer (1926) describe as the *logocentric predicament*. This his idea that logic is maximally general: any reasoning involves logic, thus a purported justification of logic would need to presuppose logic. Considering this, we could expect Frege to say that the principles of logic are only inductively confirmed, because a proof is impossible. But this does not seem to be what Frege has in mind, or else arithmetic in turn would only be inductively confirmed. Frege's commitments seem to pull him in two different directions.

To make matters more complicated, Frege scarcely discusses this issue. While the foundation of arithmetic in logic is a central topic of his formal and informal writings, he almost never discusses the epistemology of logic. There is one passage from the *Grundlagen* that interpreters tend to quote (Frege 1884, §14). I discuss that later. But the key passage that commentators focus on is this, I add line numbering for ease of commenting:

1       As to the question, why and with what right we acknowledge a logical law to be true, logic  
2       can respond only by reducing it to other logical laws. Where this is not possible, it can give  
3       no answer. Stepping outside logic, one can say: our nature and external circumstances force  
4       us to judge, and when we judge we cannot discard this law—of identity, for example—but  
5       have to acknowledge it if we do not want to lead our thinking into confusion and in the end  
6       abandon judgement altogether. I neither want to dispute nor to endorse this opinion, but  
7       merely note that what we have here is not a logical conclusion. What is offered here is not a  
8       ground of being true but of our taking to be true. And further: this impossibility, to which we  
9       are subject, of rejecting the law does not prevent us from supposing beings who do so  
10      [logical aliens];<sup>12</sup> but it does prevent us from supposing that such beings do so rightly; and it  
11      prevents us, moreover, from doubting whether it is we or they who are right. At least this is  
12      true of myself. If others dare in the same breath to both acknowledge a law and doubt it,  
13      then that seems to me to be an attempt to jump out of one's own skin against which I can  
14      only urgently warn. Whoever has once acknowledged a law of being true has thereby also  
15      acknowledged a law that prescribes what ought to be judged, wherever, whenever and by  
16      whomsoever the judgement may be made. (Frege 1893, xvii)

This is unusually evasive for Frege. In the first two lines, he seems to be saying that a justification of logic cannot be given. We are in the logocentric predicament; thus, if a justification of logic cannot be given in logic, it cannot be given at all. On line three, however, Frege suggests that one *can* step outside logic and, in doing so, one could try to justify logic by observing that we cannot discard laws

of logic. On line six, immediately after giving us this argument, Frege takes it back: the argument does not give us reason to think that the laws are true, it merely explains why we take them to be true. Moreover, this argument cannot exclude the possibility of logical aliens. On line ten, Frege gives us back the argument that he just took away. “This impossibility [humans] of rejecting the law” cannot prove that the laws of thought are true or that there can be no logical aliens, “but it prevents us from supposing that [logical aliens are right].” Frege endorses this, creating a tension with what he said on line six, and he goes on with a vivid metaphor: doubting a law of logic is an attempt to jump out of one’s skin. The last sentence is meant to illustrate this metaphor but as it turns out it is phrased quite ambiguously. Frege’s use of the verb ‘acknowledge’ there can be read in two ways: ‘if one *knows* that a law is true, then one *knows* that everyone ought to believe it’ or as ‘if one *believes* that a law is true, then one *believes* that everyone ought to believe it.’

Let me first stress what I do not take to be controversial. Frege regards laws of thought as *weakly normative*. As MacFarlane (2002) puts it: “On Frege’s view, [laws of logic and laws of physics alike] are descriptive in content but have prescriptive consequences” (37). In the pages preceding the controversial passage, Frege illustrates this very clearly:

Every law stating what is the case [descriptive] can be conceived as prescriptive, one should think in accordance with it, and in that sense it is accordingly a law of thought. This holds for geometrical and physical laws no less than for the logical (Frege 1893, xvii).<sup>13</sup>

Interpreters seem to agree that Frege’s laws of thought are not *about* how humans should think (Stroud 2018 and Conant 2020, among others). The controversial passage is not about the *content* of laws, but rather their *justification*. Having argued that logical laws are descriptive claims about the world, not prescriptions about minds, Frege asks how we are justified in believing them. Here Conant says that Frege is self-defeating, and his opponents claim that Frege takes back the question of justification he just posed.

## II.

I agree that Frege is taking something back in the controversial passage. He is a logician, and he says that the question on the justification of logic cannot be answered by logic (line two). Then he sketches a non-logical answer, but he adds that he neither disputes nor endorses this answer (line six). However, it would be too quick to conclude, as De Pierris does, that “Frege leaves our knowledge of logic and its objectivity without justification [...] because logic itself cannot give the answer” (1988, 310).<sup>14</sup>

Throughout his life, Frege is deeply concerned with the truth of the basic laws. In his reply to the letter in which Russell told him about the paradox of basic law V, Frege writes “your discovery of the contradiction [...] left me thunderstruck, because it has rocked the ground on which I meant to build arithmetic. It seems that [...] my law V is false” (Correspondence, 131). Famously, even before Russell’s discovery of this contradiction, Frege had already expressed doubts about basic law V (1883, vii). And long after that, Frege still wrote: “we cannot accept a thought as an axiom if we are in doubt about its truth” (1914, 205).<sup>15</sup>

Stroud (2018) emphasizes that one can be committed to the truth of a law and maintain that no justification can be given to it. But this attitude, which Stroud presents as his own, can hardly be attributed to Frege. This would amount to saying that Frege, for a time, thought that basic law V was true, and that the question as to how a subject can be justified in believing basic law V is unanswerable: one cannot say what would make a subject justified in believing it. But this suggests that Frege recognized no objective criteria for accepting basic law V, which is plainly false. Upon receiving Russell’s letter, Frege seems to have formed the belief that basic law V is false. Therefore, he recognized at least one necessary condition for accepting it; namely, if a thought is a basic law, then it entails no contradiction.

One could reply that this observation only shows that basic law V was not basic after all. Perhaps what Frege truly took as basic was the principle that no contradiction is true, which entails—given weak normativity as a bridge principle—that one must reject thoughts that one knows to be contradictory. I am skeptical about this, because I am inclined to take Frege at face value when he says that basic laws are basic. But let us consider this interpretation anyway: Frege gives a justification of the basic laws, to deny them would be contradictory, but he appeals to self-evidence to justify non-contradiction. Now, there is a grain of truth in the idea that Frege appeals to the principle of non-contradiction to justify some basic laws. For instance, he introduces basic law I as follows.

According to §12



Would be the False only if  $\Gamma$  and  $\Delta$  were the True while  $\Gamma$  was not the True. This is Impossible; accordingly



(Frege 1893, §18)

An instance of basic law I could be false only if  $\Gamma$  were true and not true. This is impossible; therefore, basic law I must be true. This is a *reductio ad absurdum*. But it is hardly an appeal to unexplainable self-evidence. Frege refers to §12, which is the section where he introduced the conditional stroke. There, Frege says that the value of a conditional “shall be the False if the True is taken as the [antecedent], while any object that is not the True is taken as [consequent]” (1893, §12). Now, this employs concepts that Frege introduced in the sections above. The horizontal and judgement strokes, negation, and the idea that no function can refer to both to the True and to the False are presented in sections §1 to §6, where Frege works out these notions informally with references to *Über Sinn und Bedeutung*. For instance, Frege writes “the value of the function  $\mathcal{E}^2 = 4$  is either the truth-value of the True, or that of the False [footnote: I argued for this in more detail in my essay *Über Sinn und Bedeutung*]” (1893 §2, referring to Frege 1892). The impossibility to which Frege appeals in §18—that  $\Gamma$  cannot refer to both the True and the False—is an upshot of his theory of reference. Basic law I must be true because of how Frege defined the notions at play in it, and these notions had to be defined this way to fit Frege’s theory of reference.

One could ask, then, what justifies the theory of reference itself? If in *Über Sinn und Bedeutung* Frege appealed to self-evidence, that could vindicate the self-evidence interpretation. But, in fact, Frege never appeals to self-evidence there. What he assumes are claims that he takes to reflect fully uncontroversial facts about scientific language. For instance, he assumes that in a scientific language, we demand that every term has a reference.<sup>16</sup> Frege seems to have taken for granted these claims. Never mind whether these claims are true, or justified, that would be the topic for an entirely different discussion. What matters here is that Frege did not need to postulate a notion such as self-evidence to support their basic commitments. He thought that he could support them *a priori* with his own familiarity with the human mind and scientific language.<sup>17</sup>

In general, Frege never talks in terms of self-evidence in the sections where he introduces the basic laws (1893, §1-52). On one occasion, Frege uses ‘*selbstverständlich*’ referring to the sentence  $a=a$  (§49). But here too Frege refers to his informal discussion about the equality sign, and he goes on giving a proof that  $a=a$ . Thus, this hardly supports the self-evidence interpretation. If this interpretation were correct, we would expect Frege to introduce the basic laws by claiming that they are unexplainably true. But, instead, case by case Frege gives explanations by working out informal arguments to the conclusion that the laws are true.

If we look at other parts of Frege's corpus, on the other hand, it is possible to find reasons to endorse the self-evidence interpretation. Famously, Frege uses the word '*einleuchten*' in connection to basic laws V after learning about Russell's paradox (Frege 1903, appendix 235-65).<sup>18</sup> And in a 1914 article that De Pierris quotes he wrote:

Science demands that we prove whatever is susceptible of proof and that we do not rest until we come up against something unprovable. It must endeavour to make the circle of unprovable *primitive truths* as small as possible, for the whole of mathematics is contained in these primitive truths as kernel. (Frege 1914, 204-5).

This passage can be read as saying that logical axioms are self-justifying. The problem is that this hardly fits the methodology that Frege uses to discuss his basic laws. And it hardly fits what Frege says in the controversial passage itself; he does not endorse the reasoning given "stepping outside logic" but that is reasoning nonetheless, not an appeal to self-evidence.<sup>19</sup>

I suggest explaining away this tension by distinguishing two phases of Frege's thought concerning the justification of basic laws. After he learned about Russell's paradox, Frege started mentioning self-evidence more often. And then, gradually, talk in terms of self-evidence became quite common in his writings.<sup>20</sup> This explains why interpreters defending the self-evidence reading tend to quote later works. On the other hand, if we look at what Frege wrote before learning about the Russell paradox, we find few statements that can be interpreted as appeals to self-evidence. In the *Grundlagen*, for instance, Frege says that the laws of logic "neither need nor admit proof" (1884, §3). This might be an appeal to self-evidence, but Frege might also be making the weaker claim that one cannot *formally* prove logical axioms, although one can "step outside logic" to give rational reasons to believe them. The latter reading seems to better fit what Frege writes, using German rather than his formal language, to illustrate why his basic laws cannot be false.<sup>21</sup>

There are reasons to doubt that even the later Frege endorsed the self-evidence view. This view seems to create problems with other Fregean ideas. It is natural to conceive self-evidence as a psychological notion. Surely this is how Russell (1913, 492) and Wittgenstein (WSM, 270) conceived it. On this reading, the self-evidence view makes Frege appeal to a psychological notion to justify basic laws. This does not square with Frege's rejection of psychologism, to which he was committed throughout his life. Of course, one might understand self-evidence in non-psychological terms. Dummett (1980) suggests that Frege held *epistemological platonism*. This is the view that we have an awareness, analogous to sense perception, of mind-independent objects in realm of thought.<sup>22</sup> But this only raises more questions. Russell (1912) held epistemological platonism and yet he conceded that self-evidence is a psychological notion. Therefore, this view is not automatically anti-psychologist.<sup>23</sup> Moreover, Dummett's interpretation is itself controversial. Indeed, De Pierris argues that Frege never worked out an explicit epistemological account of self-evidence. It seems, then, that if the later Frege really held the self-evidence view, he did not clearly spell-out how to reconcile it with his anti-psychologism.

In general, passages where the later Frege seems to endorse the self-evidence view tend to be brief and elusive.<sup>24</sup> Thus, it is unclear whether even the later Frege held this view. And surely, he did not hold it in the *Grundgesetze*, which is the context of the controversial passage. The question is, then, what does Frege take himself to be doing when he gives informal arguments to justify the basic laws? This leads us to Conant's interpretation, according to which Frege gave a justification for the laws of logic: their negation is not a thought, it is impossible to grasp.

### III.

Conant attributes the same view to Kant, Frege, and Wittgenstein. Laws of logic are a condition of possibility for thought: if something violates them, then it is not a thought. Kant thought that "I can think whatever I like, as long as I do not contradict myself, i.e., as long as my concept is a possible

thought” (Footnote Bxxvi). That is, the principle of non-contradiction, for Kant, is a condition of possibility for thought; one can *say* that a contradiction is true, but one cannot *think* a contradiction. In Frege’s terminology, this is to say that no thought corresponds to contradictions; thus, they are neither true nor false. One might assert a contradiction only by mistakenly associating a thought with a sentence that really has no meaning.

If Frege also held this view, then logicism as he conceived it was doomed to failure regardless of basic law V. Assume that contradictions are defective sentences to which no thought corresponds. Then, since the negation of gibberish is also gibberish, their negation is also defective. But the negation of a contradiction is a logical truth. Thus, with relatively basic assumptions, the view that logical laws define the boundaries of thought entails that logical laws lack propositional content.<sup>25</sup> This would force Frege into a contradiction, unless he rejected semantic compositionality, his theory of reference, or the idea that the basic laws are true. But these are all irreplaceable elements of Frege’s logicism!

This should not be mistaken for an argument against Conant’s interpretation. Conant is perfectly aware that his reading creates a tension between Frege’s views. He holds, though, that this is the most plausible interpretation given what Frege says. To argue against this, I need to propose an alternative interpretation and give my reasons to think that it is overall more plausible.

On lines eight and nine of the controversial passage, Frege mentions an “impossibility” to which humans are subject. And, on the previous page, he says that, should he encounter a logical alien, he would say: “here we have a hitherto unknown kind of madness” (1893, xvii). This echoes a passage from the *Grundlagen*: “we have only to try denying any [law of arithmetic], and complete confusion ensues. Even to think at all seems no longer possible” (1884, §14). Conant’s reading is that these passages say that the boundaries of logic coincide with the boundaries of thought. If we try to grasp the denial of a law of thought, confusion ensues because we are attempting to grasp something that is not a thought at all. That would be madness.

MacFarlane (2002) suggests an alternative interpretation for Frege’s claim that complete confusion ensues as soon as one tries to deny a truth of arithmetic (1884, §14). This can be adapted to explain why later Frege thought that it would be madness to take a contradiction to be true (1893, xvii). Suppose that a subject believes a thought  $P$ , and she also believes  $\sim P$ . Now, by disjunction introduction, the subject can form the belief  $P \vee Q$ , where  $Q$  is any thought whatsoever. Then by disjunction elimination the subject can prove  $Q$ . Since  $Q$  can be any thought, this shows that under Frege’s assumptions accepting a contradiction renders our standards for reasoning trivial. This is the logical principle *ex falso quodlibet*.

In this reading, confusion ensues not merely by *grasping* a contradiction, but by *asserting* it. Therefore, humans are bound to accept logical laws.<sup>26</sup> For instance, we are bound to accept basic law I because it could only be false if a thought were true and false at the same time, and this is not possible given Frege’s background theory of reference. Now, given this, should we deny any logical law, we would find ourselves believing a contradiction. That is, we would find ourselves believing both the law, because we must, and its negation. But the logical laws that we are bound to accept entail *ex falso quodlibet*. Therefore, we have only to deny any law of logic to be compelled to accept that any thought is true. Of course, one might fail to realize this. Psychological logicians, according to Frege, are exactly in this predicament. But as a matter of fact, denying any logical law implies triviality, absurdity, madness.

In short, I suggest that the impossibility that Frege mentions in the controversial passage is an impossibility of *asserting* the denial of logical laws, rather than *grasping* it. Contradictions have a sense, but humans must reject them on pain of triviality. I have two reasons to prefer this interpretation over the *grasping* interpretation.

First, this reading fits better what Frege says in the crucial passages (the controversial passage and Frege 1884, §14). On line nine, Frege clarifies that the impossibility, to which we are subject, is an impossibility “of rejecting” [zu *verwerfen*] logical laws. And, analogously, in the *Grundlagen* he says that we have only to try “denying” [leugnen] a law of arithmetic to enter a state of confusion. Now, the most natural reading for ‘rejecting’ and ‘denying’ a law seem to be asserting to its negation, rather than merely grasping it.<sup>27</sup>

Frege seems to follow this natural pattern of use. In the *Grundgesetze*, he uses ‘*verwerfen*’ sparingly: except for the controversial passage, he only rarely uses it in volume two. But when he does use it, the word is always connected to views that he takes to be false, namely Heine’s and Thomae’s views and generally, the view that “piecemeal definitions” are acceptable in mathematics (1903, §58, §60, §65, §100, and §121). Analogous considerations apply to how he uses ‘leugnen’ in the *Grundlagen*. For instance, Frege writes:

It would also be wrong to deny [leugnen] that existence and oneness can ever themselves be component characteristic of a concept. What is true is only that they are not components of those particular concepts to which language might tempt us to ascribe them. (Frege 1884, §53)

Here the word is quite clearly used in opposition to truth, rather than meaningfulness. I have no principled reason to exclude that Frege might have committed to views that entail a contradiction, as he did with basic law V. But in the case at hand this would amount to saying that where he wrote “rejecting a law” he really meant ‘grasping the negation of a law’ and this seems highly unlikely. Throughout his life, Frege emphasises the distinction between grasping a thought and asserting it. Indeed, each of his lengthier technical works begins by setting the distinction between the judgment stroke and the horizontal, or content stroke.<sup>28</sup> It seems most unlikely that Frege could have been led to a contradiction by confusing the very notions that he kept distinguishing. And this is even more evident when we consider that in the controversial passage the word ‘*verwerfen*’ occurs three times.<sup>29</sup> This suggests that Frege resolutely intended to use the verb ‘to reject’ there; thus, we should take him literally.

Second, this interpretation fits better the methodology with which Frege introduces his basic laws and justifies them informally. Consider again how Frege introduces basic law I (1893 §18). Frege invites the reader to grasp the law, by writing it with a horizontal stroke at its left. Then he observes that the law could only be false if  $\Gamma$  were true and false. This is impossible, and accordingly, Frege asserts the law by writing it with the judgement stroke at its left. The impossibility at play here is of the law being false. In no way is Frege suggesting here that he takes the negation of the law to be meaningless, rather he is saying that the law is impossible to reject, given how he defined the notions at play.

In the *Grundgesetze*, Frege never introduces a formal symbol, or informal technical terminology denoting meaningless strings of symbols. On the contrary he insists that, unlike natural language, his *Begriffsschrift* contains no names that fail to refer. Crucially, having introduced the horizontal stroke, Frege writes: “Names without reference must not occur in concept-script” (1893, §5 third footnote). This entails, given Frege’s theory of reference, that all names in the *Begriffsschrift* have a sense because a name’s sense is the way in which it displays its reference. And this, in turn, entails that all sentences in the *Begriffsschrift* have a sense, because sentences are names of the True or the False. If Frege truly intended to justify the basic laws by saying that their negation is meaningless, he did so rather cryptically, since his technical vocabulary does not allow him to say such a thing.

Admittedly, one could reply that when Frege says that to justify the basic laws we would need to “step outside logic” (line three), he intends to gesture at a notion that his technical vocabulary cannot refer to. Perhaps Frege thought that although he cannot say that the negation of basic law I is meaningless, he can show it. But, again, there is an alternative interpretation. In the introduction

to the *Begriffsschrift*, Frege famously compares his formal language with a microscope and natural languages with the human eye (1879, 105). The former is better suited for specific scientific purposes whereas the latter is much more versatile. This helps explaining how we can step outside logic even though we are in the logocentric predicament. We are in the logocentric predicament since we are bound to take the basic laws to be true. But we step outside logic whenever we reason using natural language rather than a formal language. Thus, Frege is stepping outside logic every time he writes in German. For instance, when he introduces basic law I he steps outside logic to explain, in German, that the law could not be false.

This interpretation takes at face value the controversial passage without attributing to Frege a very cryptic view. Frege thought that, due to the expressive limitations of his *Begriffsschrift*, it is sometimes necessary to formulate his views informally. This is all we need to illuminate the metaphor of stepping outside logic. We hardly need an extra interpretative layer where Frege *shows* that negated basic laws are meaningless even though he lacks vocabulary to *say* that.

#### IV.

I started with the observation that, *prima facie*, the view that I attribute to Frege seems both wrong and unsatisfactory. It seems wrong because, plainly, people can make mistakes in logic; a classical logician may endorse a principle that entails contradiction, as Frege did. It seems unsatisfactory because, even if it were impossible for us to deny the basic laws, it would still be unclear how this justifies them. In this last section, I discuss how Frege meant to address these issues.

Even before discovering that basic law V leads to a contradiction, Frege was perfectly aware that people can make logical mistakes. Indeed, as I pointed out above, he thought that psychologist logicians are in this predicament: they fail to realize that their view leads to absurdity, but in fact it does. People can make logical mistakes in at least two senses. On the one hand, we can use languages that are not suitable for rigorous scientific reasoning. Frege acknowledges that terms in natural language might fail to refer (1892, 40).<sup>30</sup> On the other hand, we can make logical mistakes in the sense that, even using a formal language, we may fail to see all the consequences of the principles we commit ourselves to.

To reject a basic law, according to Frege, is *ideally* impossible. That is, it is impossible if we disregard both these sources of error: an agent cannot reject a basic law when (i) she uses a language ideally suitable for scientific enquiry and (ii) she can see all the consequence of the principles she commits herself to. This, I suggest, is the “impossibility” on line eight. Now, the second condition is the familiar notion of an ideal calculator. The first condition, on the other hand, requires further discussion.

On lines three to six, Frege considers an argument that appeals to the ideal impossibility to reject basic laws and concludes that they are justified. Then he expresses scepticism about this argument for two reasons. First, the argument shows only that it is ideally impossible to reject the basic laws, it hardly shows that they are in fact true (lines seven and eight). Second, we cannot exclude the possibility of logical aliens (line nine).

This should certainly give us pause. As we saw, Frege introduces basic law I by pointing out that it cannot be false (Frege 1893, §18). But then, in the controversial passage, he seems to make the weaker claim that we cannot *take it* to be false. This creates a tension. If Frege had a stronger argument showing that basic laws cannot be false, why would he refrain from using it in the controversial passage? And, on the other hand, if he lacked such a strong argument, as the controversial passage suggests, how could he introduce the basic laws by claiming that they cannot be false? The first condition for ideal reasoning resolves this tension.

When he introduces basic law I, Frege argues that it cannot be false *given how he defined the notions in it*. Namely, he appeals to the stipulations he made in §12 concerning the conditional stroke. Under different stipulations the same formulas could be false.<sup>31</sup>

Frege justifies his stipulations by referring to his theory of reference. There, he argues that when language is used for scientific purposes it must meet these standards, a claim which motivates his stipulations. Now, the evidence supporting Frege's understanding of scientific language seems to be his own familiarity with scientific languages and the human mind. He takes for granted claims that, according to him, reflect uncontroversial facts about scientific discourse. This makes the justification of basic laws somewhat human-centred. The laws cannot be false *in the Begriffsschrift*, which is an ideal language for human scientific discourse. But a different community might not accept the same claims about scientific language that we accept; for instance, a community might not accept that in scientific language no term should fail to refer.

Frege repeatedly envisions the possibility of mathematical aliens. He seems to take this as a nomological possibility, since he considers scenarios in which, because of evolution, humans could come to believe that  $2 \times 2 = 3$  or  $2 \times 2 = 5$  (1884, xviii-xix; 1906b, 175). He always emphasises that arithmetical truths, and logical, are ontologically independent from human beliefs. But here he is asking how we can justify our mathematical and logical beliefs. Having agreed that some unspecified truths of arithmetic are mind-independent, the question remains open as to whether we are right to think that, say, the sentence ' $2 \times 2 = 4$ ' corresponds to a truth of arithmetic, rather than ' $2 \times 2 = 3$ .' Frege attempts to show that ' $2 \times 2 = 4$ ' must be true in a language that meets our basic scientific standards. But he also admits that this only proves that we must take it to be true. It hardly proves that ' $2 \times 2 = 4$ ' in fact corresponds to a mathematical truth, let alone showing that logical aliens are impossible.

Frege's reason to believe the basic laws, even though logical aliens are possible is this: the claim that ideal humans must assert laws that quantify over any object whatsoever, paired with the rejection of skepticism, leads to the conclusion that said laws are true. Suppose that ideal humans can never assert  $\sim P \rightarrow P$ , but in fact some particular sentences fail to entail themselves. Ideal humans could never identify the anomalous cases, since as far as they are concerned, the basic laws are universally valid. This casts a doubt over all their beliefs. Anything at all that is given to us in scientific belief is subject to the basic laws, and to the laws of mathematics. Therefore, in Frege's framework, doubting the basic laws amounts to doubting the human epistemic standpoint as a whole.

On lines ten and eleven, Frege observes that although we cannot exclude the possibility of logical aliens, we are unable to suppose that they might be right. This is a consequence of the weak normativity of descriptive laws. If one accepts a law, one thereby accepts that everybody ought to accept it (lines fifteen and sixteen). Therefore, since ideal humans are bound to assert laws of logic, we, non-ideal humans, may conclude that logical aliens believe something that they ought not to. Unlike ideal calculators, we can mistake a contradiction for a law. That is, we might assert something that we would realize we ought not to, if we reasoned unimpeded by psychological limitations. Yet, we know that logical aliens are wrong because, by definition, they reject laws that ideal humans accept, whatever these laws might turn out to be. This is still human-centric. By the reasoning above, we cannot prove that logical aliens are wrong, but we are bound to take them to be wrong. And, conversely, *they* believe that we are mistaken. We are both stuck in our epistemic standpoints.

Now, we might be unable to rule out the possibility that our epistemic standpoint is grossly mistaken. But, says Frege speaking for himself now, questioning our epistemic standpoint is unadvisable; it would be as unadvisable as to try jumping out of one's skin (lines thirteen and fourteen). If we are to do science, we better assume that there is nothing preventing us in principle from accumulating knowledge. Rather than attempting a refutation of skepticism of the sort that Kant tried to produce time and time again,<sup>32</sup> Frege suggests disregarding skepticism.

One could argue that this solution falls short of a justification of logic. Indeed, this is not the sort of proof that Kant hoped to give in his “refutation of idealism” (B274-9). But this is so because Frege is much more aware that some arguments can only be given after ‘stepping outside logic.’ As far as informal arguments go, Frege’s reasoning in the controversial passage is sound. Skepticism is a scientific dead end. Therefore, to engage in scientific investigation, we must set it aside. If others dare doing otherwise, says Frege, one can only urgently warn them. This is not a proof, but it gives us a strong motivation to accept Frege’s basic laws. Either we accept them, or we claim that there is no way humans can, even ideally, accumulate knowledge.<sup>33</sup>

## Notes

<sup>1</sup> Unless otherwise noted, all quotations from the *Grundgesetze* are to the Ebert and Rossberg (2016) translation. I only discuss translation issues in notes (see notes 6, 24, and 29)

<sup>2</sup> As Conant (1992) emphasizes, the idea that logical mistakes are in some sense impossible can be found in the early Wittgenstein’s writings (TLP, 5.473, 5.4731; NB, 2) I discuss elsewhere how Frege’s justification of logic relates to Wittgenstein’s. For the moment I should only say that, in my interpretation of Frege, logical mistakes are not impossible in the sense that Conant attributes to both Frege and Wittgenstein.

<sup>3</sup> A third important problem is this: in a sense, Frege’s logical aliens are now commonplace; thus, they are hardly impossible. For instance, Priest (1987) believes in true contradictions. This is a possibility that Frege never considered, because he never encountered, or envisioned, a logician seriously defending non-classical logics. To apply Frege’s argument to the contemporary debate one would have to take this into account. But this is not relevant for the present historical discussion. Thus, I discuss this elsewhere.

<sup>4</sup> See also TLP, 6.1271 5.4731.

<sup>5</sup> See also Hacker 1986, 43, among others.

<sup>6</sup> When it comes to interpreting Frege’s epistemology of logic, interpretative issues overlap with translation issues. De Pierris translates ‘*einleuchten*’ as ‘self-evident,’ just like Geach does (Frege 1960, 234). Ebert and Rossberg translate it as ‘obvious’ in the same passage (Frege 2016, 253) because, they say: “Frege uses the term to indicate that something, such as a proof, is easily understood or accepted” (Ebert & Rossberg 2016, xxii). In TLP 6.1271, where Wittgenstein criticizes Frege’s alleged appeal to self-evidence, he uses the term ‘*einleuchten*.’ This might suggest that he was targeting the notion that De Pierris and Geach translate as ‘self-evident.’ But then it is surprising that Frege does not use it in the passages usually quoted in connection with the secondary problem of pure reason (Frege 1884, §14; 1893, xvii). Nor does he use ‘*selbstverständlich*,’ the word that Ebert and Rossberg too translate as ‘self-evident.’ I maintain that translation issues can be set aside: regardless of translation, if Frege intended to claim that the basic laws are self-evident, he would have used at least one of the words (*einleuchten* or *selbstverständlich*) in the crucial passages.

<sup>7</sup> This idea is also in Friedman 1985 (endnote 72).

<sup>8</sup> Conant and Diamond (2004) argue, contra Williams (2004), that TLP is in no sense contradictory or self-defeating. Therefore, in Conant’s reading, Wittgenstein is overall less likely to have a self-defeating lapsus than Frege. I argue elsewhere that it is TLP that seems to contain lapsus in connection to the justification of logic.

<sup>9</sup> See also, among others, Frege 1906c, 184 and 1906d, 185.

<sup>10</sup> Strictly speaking this is incorrect. To complete his project, Frege only needed to derive the Peano axioms, or other axioms from which the rest of arithmetic is derivable. He says otherwise because at the time this had not yet clearly emerged.

<sup>11</sup> I say that this is Frege’s *lifelong* project because this idea recurs throughout Frege’s writing. In the *Begriffsschrift*, Frege says that he aims to see “how far one could get in arithmetic [...] supported only by the laws of thought, which transcend all particulars” and he adds, making it quite clear that he already has in mind his logicist project, that he developed his conceptual notation to make sure that, in the deduction of arithmetic, “something intuitive could not squeeze in unnoticed” (Frege 1879, 104).

<sup>12</sup> The term “logical aliens” was used by Conant (1992), whose title “In search for logically alien thought” seems to refer back to Sluga’s (1980) chapter V “In search of logical objects” (96-123). The term is meant to refer to subjects that reason in a way that humans cannot possibly reason. This is somewhat ambiguous: a logical alien may be (i) a subject that can grasp unthinkable thoughts, (ii) a subject that can *grasp* thoughts that humans

cannot grasp, (iii) a subject that can *assert* thoughts that humans cannot assert, or (iv) a subject that can *grasp or assert* thoughts that humans cannot grasp or assert. Now, definition (i) is self-defeating if by ‘unthinkable thought’ we mean a mock-proposition that cannot be grasped. Thus, definition (i) should not be adopted from the outset; perhaps Frege thought that logical aliens are impossible because humans can grasp all thinkable thoughts, but this should result from an argument of his, not from how we define the terms in our interpretation of his work. Definitions (ii) and (iii) are biased in favour of Conant’s and my reading respectively. Conant suggests that the impossibility on line eight of the controversial passage amounts to an impossibility of *grasping* certain thoughts, I argue that it amounts to an impossibility of *asserting* certain thoughts. The definition of the terms we use to discuss these readings should presuppose neither. Therefore, I adopt definition (iv), which entails that Conant and I can agree that the controversial passage involve logical aliens, but we disagree on which aspect of logical alien-hood is at play. An equivalent, alternative rendering of this disagreement presupposes definitions (ii) or (iii) instead, which entails that Conant or I maintain, contra the other, that the controversial passage does not involve logical aliens.

<sup>13</sup> See also Frege 1918, 351.

<sup>14</sup> De Pierris only quotes the first half of the controversial passage, just like Hacker and Stroud do, among others (Stroud 2018; Hacker 1986, 43).

<sup>15</sup> By 1906 Frege gave up hope of rescuing basic law V from Russell’s paradox (see Frege 1906a). Thus, Frege seems concerned with the truth of the laws of logic independently of the prospects of his logicist project.

<sup>16</sup> I say *scientific* language because Frege sets aside cases in which we use language for “aesthetic delight” in favour of “an attitude of scientific investigation” in which we demand that no term fails to refer (Frege 1892, 33). On this ground, Frege postulates that no terms in his formal language fail to refer, since it is a language designed for science (Frege 1893, §5 third footnote).

<sup>17</sup> It might seem psychologistic of Frege to appeal to his knowledge of his own mind. But Frege thought that only the study of individual minds belongs to psychology. Frege writes:

Nothing would be a greater misunderstanding of mathematics than making it subordinate to psychology. Neither logic nor mathematics has the task of investigating minds and contents of consciousness owned by individual men [sic.]. Their task could perhaps be represented rather as the investigation of *the* mind; of *the* mind, not of minds. (Frege 1918, 368-9)

This strongly suggests that Frege is reasoning about the thought processes and language of ideal humans, to which he takes himself to have a direct access, albeit a fallible access.

<sup>18</sup> See also Frege 1906a where Frege writes that Schoenflies’s version of basic law V is not “as self-evident as one would wish for a law of logic” (182).

<sup>19</sup> Notice also that the correspondence with Russell concerning basic law V is hardly a discussion on whether basic law V is self-evident.

<sup>20</sup> Incidentally, Russell seems to have had a similar, but more explicit, change of opinion. In *Principia Mathematica* Russell and Whitehead maintain that self-evidence is not indispensable to knowledge (Russell & Whitehead 1910, 59). But later Russell explicitly commits to the self-evidence view (Russell 1913, 492)

<sup>21</sup> Another passage where the early Frege might seem to endorse the self-evidence view is this

In arithmetic [...] we must demand that any assertion that is not completely self-evident [*selbstverständlich*] must actually be proven (1903, §60).

Out of context, this might seem a claim about the justification of basic laws. But, in fact, here Frege is criticising the habit of some mathematicians of omitting important steps in their proof and giving circular or vague definitions. Frege is not talking about basic laws here. Accordingly, it seems difficult to read this as a passage about the justification of basic laws. More likely, Frege is making the point that one must omit no step of one’s proofs, unless it is absolutely clear what the missing step would be, as he sometimes does, for instance, omitting explicit steps to remove redundant parentheses. The word ‘*selbstverständlich*’ appears as well in the previous section. Here even out of context the self-evidence interpretation seems off. Frege writes:

It should strike one as entirely self-evident that a word should not be explained by means of itself, for otherwise it is treated in one breath as both known and unknown (1903, §60).

What is said to be self-evident here is the claim that one should give no circular definitions, which is hardly a basic law. And even in this case self-evidence is not a standalone epistemic concept, as Frege’s later writing might seem to suggest, because Frege gives a reason why one should give no circular definition, the reason being that circular definitions contradictorily entail that the definiens is both known and unknown.

<sup>22</sup> This also fails to fit what Frege wrote in the *Grundgesetze*. Nowhere there does Frege say that he can perceive the basic laws in the third realm. Therefore, Dummett's is also an interpretation of the later Frege.

<sup>23</sup> *Ontological platonism* is the view that logical laws are true in virtue of properties of mind-independent entities populating the realm of thought. This grants that the truth of the laws does not ontologically depend on mental facts. In "Thought" Frege explicitly commits himself to this (1918, 363). But ontological platonism is compatible with a psychologist epistemology. Crucially, it seems natural to admit, for epistemological platonists, that our awareness of the third realm is biased by our mental makeup. And this in turn makes the commitment to ontological platonism seem a *petitio principii*: logical laws are ontologically mind independent, but whatever law one might accept is justified on psychological grounds. This is the sort of tension that Wittgenstein alludes to when he criticizes Frege's alleged appeal to self-evidence. Now, I am not suggesting that this issue cannot be solved by a properly worked out platonism, but rather that it is unclear how Frege meant to solve this, if he held epistemic platonism.

<sup>24</sup> Consider Geach's translation of the famous sentence in the appendix to the *Grundgesetze*: "I have never disguised myself from [Axiom V's] lack of self-evidence" (1960, 234). As Ebert and Rossberg observe, the word '*einleuchten*' is not a technical epistemological term in German, they translate it as 'obvious' instead. And even in Geach's translation, this is hardly conclusive evidence for the self-evidence interpretation. Rather, the passage suggests that, for Frege, the feeling that a thought is self-evident is an indicator that it could be a law, but there are other more reliable markers; hence, for a time he thought that axiom V was a basic law although he always felt that it is not self-evident. De Pierris's quote from Frege (1914) is similarly ambiguous. There, Frege might be making the claim that the primitive truths of arithmetic cannot be *formally* proven except in the trivial sense in which any axiom is a theorem, this is compatible with the idea that they can be given an informal justification. The De Pierris's quote suggests the self-evidence reading because Frege says that logical axioms are *primitive* truths (1914, 205). But then in the next page he goes on to say that two equivalent theories can have different axioms. Thus, the notion of *primitivity* at play in the article does not involve specific laws. Frege might very well just be making the now trivial point that not every truth in a logical theory can be non-circularly proven, which is hardly a point about the justification of logic. One could expect to find Frege's later view in a 1924-25 manuscript "Sources of Knowledge of Mathematics and the mathematical natural Sciences." But unfortunately this piece makes no mention of self-evidence. Indeed, it seems hard to distill *any* view on the justification of the basic laws from this manuscript.

<sup>25</sup> Indeed, Kant seems compelled to hold that we can only have procedural knowledge or understanding of logical laws, because they lack what we may now call propositional content. Kant, if only implicitly, distinguishes procedural and declarative knowledge. A logician, for Kant, knows how to discern bogus sentences from thoughts but she possesses no propositional knowledge about logic.

<sup>26</sup> Notice that here Frege is presupposing that if a sentence P cannot be asserted, then its negation must be. This presupposes a now controversial view about negation (Frege 1919; Rumfit 2000; Ripley 2011).

<sup>27</sup> I would like to thank [blinded for review] for making this point during our discussion concerning the *Grundlagen* passage, and for pointing out that the claim that contradictions are meaningless, under Frege's assumptions, entails that logical truths are meaningless too.

<sup>28</sup> See also Frege 1906c, 184 and 1906d, 185.

<sup>29</sup> The repetition of '*verwerfen*' is hidden in the translation. Ebert and Rossberg translate the word as 'discard' on line four of the controversial passage. And, on line nine, where Frege uses it again, they edited it out writing "beings who do so" rather than "beings who reject it" [*die es verwerfen*]. This does not create a loss in meaning, since in the context 'reject' and 'discard' convey analogous meanings. But it should be noted that in the controversial passage Frege uses three times a word that he otherwise uses sparingly. This creates an emphasis that is lost in the translation, although literal meaning is preserved.

<sup>30</sup> Ever since he wrote the *Begriffsschrift*, Frege regarded his formal language as a tool for an ideally rigorous scientific reasoning. In the end of the introduction to the *Begriffsschrift*, Frege suggests that his formal language could be extended to express geometrical and physical notions as well, and he compares his project to Leibniz's idea of a *calculus philosophicus* (1884, 107). Thus, he envisioned the possibility of extending the ideal thinking in his concept script to other scientific disciplines. But certainly he thought that this was restricted to scientific thought, given the microscope analogy.

<sup>31</sup> For instance, should we remove the assumption that no term can fail to refer, the truth-table of the conditional stroke would change because it would have to account not only for the cases in which antecedent

and consequent are true or false, but also for the cases in which they fail to refer. This could affect basic law I since Frege appeals to the definition of the conditional stroke to justify it.

<sup>32</sup> There are a number of different versions of Kant's argument against skepticism. This is presented in "the refutation of idealism" (B274-9). Then in the B preface Kant recommends a first amendment (footnote Bxxxix), and then he recommends a second amendment in the "General Note on the System of Principles" (B288-94). Furthermore, after the publication of the second edition, Kant drafted new versions of that argument, including R 5654 (18:305-13) and R 6311 (18:607). Guyer (1983) discusses the various version of this argument, arguing that Kant intended to establish our knowledge about *ontologically*, rather than *phenomenologically* external objects.

<sup>33</sup> See also Frege 1918 where again Frege seems to treat entailing the actuality of a skeptical scenario as a *reductio ad absurdum*. He argues that a view he wants to deny entails skepticism and on that grounds, he discards it, the view being that mind-dependent ideas are the only possible objects of human awareness (363-8). Notably, the final piece of evidence that Frege gives is that, if we take seriously skepticism, science collapses: "the natural sciences too could only be assessed as fables like astrology and alchemy" (368).

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