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COUNTERFACTUALS WITHOUT POSSIBLE WORLDS

1. INTRODUCTION

In his *Fact, Fiction and Forecast* Nelson Goodman proposed that a counterfactual

(1) "If it had been the case that ϕ then it would have been the case that ψ."

is true exactly when ψ is derivable from ϕ together with a suitable set of sentences S. Intuitively, S is to consist of sentences which (i) are in fact true and (ii) would also have been true, if contrary to fact, ϕ had been true. The second condition, which is strictly speaking a condition on S as a whole, Goodman referred to as the *contenability* of S with ϕ. The problem with (ii) as it stands is that it involves a counterfactual construction; any such account for the truth-conditions of counterfactuals must, therefore, be circular. Furthermore, there does not appear to be a simple alternative characterisation of cotenability which is free from counterfactual constructions. Goodman himself thought that the only possible way to escape from this circularity would be to account for the truth-conditions of counterfactuals in terms of underlying natural laws; and a theory of lawlikeness which would include a solution of his so-called projection problem. Unfortunately, as Goodman himself emphasises, this would destroy his original claim to explicate the concept of law by reference to counterfactuals. Worse still, it now appears that the projection problem, which Goodman subsequently came to see as the kernel of the problem of lawlikeness, can in its turn only be solved if we appeal to certain laws.

Bob Stalnaker and David Lewis independently developed theories which, although they do not solve completely the problem of truth-conditions, say enough about it to clarify to a large extent the logic of conditionals. Stalnaker supposes that the circumstances in which we use a conditional (1) determine a function (f, say) which maps sentences onto alternative situations in which the sentences are true. The conditional (1) is true in

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the situation given, exactly when in the antecedent situation determined by the function the consequent is true. Very little is said, in Stalnaker's first write-up on conditionals, about how the function f is determined. In this respect the theory of Lewis is a considerable improvement. This theory rests upon the idea that when we evaluate (1) in the given situation S we are led to consider just those alternative situations or situation in which the antecedent is true while yet as many aspects of S are retained as possible. Like Goodman's notion of cotenability this idea derives directly from the incontrovertible intuition that, in scientific as well as ordinary discourse, the truth-conditions of counterfactuals involve a *ceteris paribus* rider: here the *cetera paria* are what the situation of evaluation and the relevant alternatives have in common. Lewis determines the situations relevant to the evaluation of (1) in S in terms of a relation of comparative similarity: the relevant situations are, roughly speaking, those which are as similar to S as possible given the truth of the antecedent.

Apart from being conceptually more satisfying than Stalnaker's theory, Lewis' also has the laudable advantage of allowing for the distinction between (1) and

(2) If it had been the case that ϕ it might have been the case that ψ.

In Stalnaker's original theory there is no room for this distinction since the function f always determines a unique alternative. Lewis, however, refrains from assuming that the *ceteris paribus* problem admits of a unique solution. In other words, given S and ϕ, there may be several situations in which ϕ holds and which are equally similar to S. If in some of these ψ is true and in some others ψ is false then both (1) and

(3) If it had been the case that ϕ then it would have been the case that $\sim \psi$.

are false but (2) is true.[1]

The concept of comparative similarity does not lead to a unique concept of validity for sentences containing counterfactual conditionals. But as with the notion of an accessibility relation between possible worlds in modal logic, it provides us with a family of clearly articulatable concepts of validity, distinguished from each other by the sets of conditions that are imposed upon the alternativeness relation. Lewis' book [15], as well as his

subsequent writings [14], [16], show how fruitful his theory is in this regard.

Nevertheless, his theory has been much under attack. The majority of these objections belong under two general categories. In the first place there are those who reject the concept of 'possible worlds' as Lewis uses it – Lewis himself refers to worlds, rather than situations, as I have done so far. Then secondly, there are those who do not object to the use of possible worlds as such but who cannot assign a clear meaning to the comparative similarity relation. These two types of objections may, at first sight, seem quite unrelated. One can challenge this similarity relation without objecting to the objects among which this relation is supposed to hold. On the other hand, for one who rejects possible worlds altogether there is not even a framework within which the second objection could be considered.

I hope to show that these questions are very closely connected. Careful reflection on the concept of possible world, and its role in philosophical analysis and semantics will lead also to a better understanding of the similarity relation. Indeed, it will lead us back to a perspective of counterfactuals that in some ways resembles the original intuitions of Goodman which led to his notion of cotenability. Not that we will be able to solve the problem of cotenability completely. It appears that this cannot be done in the general setting which this paper shares with the works so far mentioned. In order to say something more substantial about the *extension* of the similarity relation one must, it seems, restrict ones attention to particular languages (e.g., the language of a particular branch of physics), and/or specific contexts of use.

Perhaps the single most convincing aspect of Lewis' theory is the treatment of the so-called 'paradoxes of conditionals'. These inference patterns were marked as valid by earlier theories despite the fact that there are obvious counter-examples. For example, consider the following pattern:[2]

$$(4) \qquad \frac{\phi \mathrel{\Box\!\!\rightarrow} \psi}{(\phi \,\&\, \delta) \mathrel{\Box\!\!\rightarrow} \psi}$$

In the following examples it is surely possible to hold (5) true and (6) false. In particular, one could claim that if the U.S.A. dumped its weapons there would be war but this would be avoided if the other nations followed suit.

(5) If the U.S.A. threw all its weapons into the sea tomorrow there would be war.

(6) If the U.S.A. threw all its weapons into the sea tomorrow and the other nations did so also there would be war.

Lewis explains the possibility of such counterexamples to (4) by observing that the worlds where the U.S.A. disposes of its weapons are *more* similar to the actual world than those where all the nations do so.

In an example like this our intuitions regarding comparative similarity of the relevant worlds seem clear enough. But what can we say generally about the principles by which we may determine the relative closeness of possible worlds? As soon as we consider this question we are confronted by our limited grasp of the nature of possible worlds. We can never command more than a partial description of any 'possible world'. It is clear, therefore, that what enables *us* to decide on questions of relative closeness must also pertain to such descriptions.

It might be argued that this 'comparative similarity' relation between these partial descriptions derives from an underlying relation between the worlds themselves. But if this were so we could no more explain how we *recognise* that two partial descriptions stand in the relation (when they do, that is). It seems more natural (and indeed necessary if we wish to account for how such matters are decided), therefore, to regard the relation between partial descriptions as sui-generis, and to attempt to explicate the notion of comparative similarity between worlds in terms of it. However, we will provisionally assume that the relation between possible worlds is given, and ask what connection there ought to be between this relation and a similarity relation between partial descriptions.

To cast some light on this question let us consider some specific examples.

(7) If Jones had been married in his twenties he might have had a daughter.

According to Lewis this is true because some of the worlds nearest to the actual world in which Jones married in his youth are worlds where he is blessed with a daughter. In other words, worlds in which he is married young and has a daughter are at least as near to our world as those in which he is married in his youth but has no daughter. This does not mean each world in which he is married young and has a daughter are at least as near as any in which he is just married (without a daughter). There surely are worlds in which he is married and has a daughter but which in other ways

are so bizarre that all in all they must be regarded as vastly more dissimilar from the actual world than at least some worlds in which Jones is married without having a daughter. These bizarre aspects may have nothing to do with Jones. In fact, in the vast majority of cases they will have nothing whatever to do with him or his marriage.

We can only maintain that *some* worlds in which Jones is married and has a daughter are at least as similar to ours as any in which he is *just* married. But what exactly is meant by this claim?

To answer this question, we must take the talk about 'possible worlds' as no more than a metaphor. What then is this last claim a metaphor for? I suggest that the idea behind the metaphor can be paraphrased as follows. Consider the descriptions

(8) 'is a world in which Jones is married and has a daughter'

(9) 'is a world in which Jones is married and has no daughter'.

Each of these descriptions can be elaborated in innumerable ways; there is no end to the various details – most of which are irrelevant to the conditional under consideration – that we could consistently add to these descriptions. Now (8) is at least as close to the actual world as (9) insofar as it is possible to match any consistent elaboration (9′) of (9) by a consistent elaboration (8′) of (8) which is at least as close to the actual world. Admittedly this principle looks circular. It does not explain the 'closer' relation in *other* terms; it only states a regularity that the notion must obey. Yet, I hope, the intuition behind the principle is clear: no matter how we extend the description (9), so as to eventually arrive at a completer world description, the successive steps in this process can always be matched by successive elaborations of (8) in such a manner that at each stage the elaboration of (8) is true of certain worlds that are at least as near to our world as any of the worlds compatible with the elaboration of (9). Moreover, and this is important, in concrete examples such as the one under consideration, our *recognition* that some worlds where Jones is married and has a daughter are at least as near our world as any worlds where he is only married seems to be based on precisely this intuition: that whatever further refinements are demanded of our presentations of these various worlds, I can always counter an elaboration of (9) (9′, say) by an elaboration (8′) of (8), which is of the same degree of detail as (9′), and which partially

describes a world which is at least as close to ours as that described by the elaboration of (9). Of course 'to describe a world which is at least as close to ours . . .' must itself be understood in terms of matching still more detailed elaborations of (9) by more detailed elaborations of (8).

It must by now be evident that I do not want to rely upon possible worlds and their similarity relation as primitive in my analysis of counterfactuals. Rather, I will base my analysis on finite/partial world descriptions and the relations of comparative similarity between them – although under certain natural assumptions concerning these world descriptions, the two accounts will turn out to be formally equivalent.

But how do we actually assess a counterfactual conditional? According to Goodman we *select* those propositions cotenable with the antecedent and which together with the latter enable us to derive (in some appropriate sense) the consequent.[3] The difficult problem is obviously the selection process; how exactly are we to think about it? The following procedure seems to me to be rather natural. We think of a number of alternative circumstances in which the antecedent could be (or could have been as the case may be) true. Some of these (in most cases only partially specified) we may regard as more plausible/probable than certain others. Suppose for instance that of the circumstances $s_1, \ldots, s_n$ considered, each of $s_1, \ldots, s_k$ is more plausible than each of the others, where $k \leqslant n$. If each of the $s_i, i \leqslant k$, together with the antecedent, imply the consequent then we may regard the conditional as true; if one of the s_i implies the negation of the antecedent then we may certainly regard the conditional as false. If neither of these cases apply we must partition the antecedent situations more finely and repeat the process.

2. CONSTRUCTIVE ALGEBRAS

What are we to say about the language in which to express our partial descriptions of possible worlds? Presumably, we must demand that the language is closed under truth-functions; but we require more than this for we need to make precise the phrase "degree of detail" which occurs in our informal analysis of comparative similarity. There are, of course, many possible ways of achieving this. I shall concentrate on one approach; it is in essence due to Hintikka ([9], [10]) and works only for first-order languages.

Hintikka observed that the sentences of any first-order predicate language

L can be naturally ordered in terms of the complexity of their quantifier structure: the greater that complexity the greater the variety of propositions expressible. Furthermore, unless L has only monadic predicates, there is for each (non-contradictory) sentence s with degree of complexity k a sentence s' of degree $k+1$ which logically entails s but is not entailed by it. Perhaps the most important contribution of Hintikka's work in this area is the following: given a sentence of surface complexity degree k ($\geqslant 1$) we can rewrite this as a sentence[4] about the existence or non-existence of individuals with properties that are expressible by formulae of degree $k-1$.[5] In particular, if s is of degree k, s' of degree $k+1$ and s' entails s then s' amounts to an assertion about the existence or non-existence of individuals with properties expressible by formulae of degree k; and s' entails s precisely because these 'properties of degree k' imply or exclude the 'properties of degree $k-1$' which figure in the normal form of s. In particular, if s asserts, among other things perhaps, the existence of an individual with a property P of degree $k-1$ then s' will contain the assertion that there is an individual with property P' of degree k and which entails, but in general is not entailed by, P. It is for such reasons that s' can be regarded as giving more information than s.[6]

To make these remarks a little more concrete we consider a very simple first-order language. For instance assume that L has just one binary relation R. Using one level of quantifiers we can state the existence of only two individuals – by the sentences $\exists xRxx$, $\exists x \sim Rxx$. With two levels of quantifiers we can describe an individual by (for instance) the sentence

$$\begin{aligned}\exists x[&\exists y(Rxy \,\&\, Ryx \,\&\, Ryy) \,\&\sim \exists y(Rxy \,\&\sim Ryx \,\&\, Ryy)\,\&\\ &\exists y(\sim Rxy \,\&\, Ryx \,\&\, Ryy) \,\&\sim \exists y(\sim Rxy \,\&\sim Ryx \,\&\\ &Ryy) \,\&\sim \exists y(Rxy \,\&\, Ryx \,\&\, Ryy) \,\&\sim \exists y(Rxy \,\&\\ &\sim Ryx \,\&\sim Ryy) \,\&\sim \exists y(\sim Rxy \,\&\, Ryx \,\&\sim Ryy) \,\&\\ &\exists y(\sim Rxy \,\&\sim Ryx \,\&\sim Ryy) \,\&\, Rxx].\end{aligned}$$

We will not strain the reader's patience by writing out all the individuals which can be defined using two levels of quantifiers; we have done enough to illustrate our previous remark; the first sentence asserts the existence of an individual with a property of higher degree which entails the property described by the first sentence. If we take the conjunction of all sentences of the form of the second sentence (or their negations) then we have a total picture of all the individuals which can be said to exist in L using two levels of quantifiers – Hintikka refers to such a conjunction as a 'constituent'.

In view of our preceding remarks, it would appear to be reasonable to identify the degree of depth or detail, of such a description with the quantifier complexity of the sentence, or sentences, which occur in it. Although I wish to present my theory in a fairly general setting, which does not depend on the particular explication of the concept of degree of detail, I shall retain in my exposition the following structural conditions which are illustrated by this example. First, only a finite number of descriptions are possible at each degree of detail – this ensures that our decision process for comparative similarity is realistic. Next, if s is a constituent of degree k and s' a constituent of degree $k+1$ then we demand that s' decides s, i.e., $s' \models s$ or $s' \models \sim s$ (and not vice-versa).

I shall base my exposition upon a more abstract representation of the language in which we suppose the partial descriptions to be given; in fact we shall assume our partial descriptions are the elements of a Boolean algebra – this will satisfy our demand of closure under truth-functions. If we wish to retain the intuitions of the more concrete approach we might consider the algebra as the Lindenbaum algebra of the first order language L (together with its proof-theory). Let $\mathscr{B} = \langle B, \cup, \cap, 0, 1, * \rangle$ be such an algebra: $\cup, \cap, *$ are respectively the union, intersection and complementation operations of the algebra where as 0, 1 are the zero and identity elements; $a \leqslant b$ is defined to be true exactly when $a \cap b^* = 0$.

In the context of Boolean algebras we can summarise our constraints on the degree of detail of a given partial description, in terms of the following concept of measure.

DEFINITION 2.1. Let $\mathscr{B}$ be a Boolean algebra. A *Measure* for $\mathscr{B}$ is any function

$$d: B \to \text{Natural Numbers}$$

such that

(i) $B_n = \{a \in B: d(a) \leqslant n\}$ is a finite subalgebra of B;

(ii) if a is an atom in B_{n+1} and b is an atom in B_n then a decides b, i.e., $a \leqslant b \vee a \leqslant b^*$.

This definition requires a word of explanation.[7] An element a of an algebra is an *atom* if $a \neq 0$ and if b is in the algebra with $b \leqslant a$ then $b = 0$ or $b = a$; an algebra is *atomic* provided that corresponding to every element

$b \neq 0$ there is an atom $a \leqslant b$ – the atoms play the role constituents played in our example. Notice that condition (ii) makes perfect sense since B_n is finite and finite algebras are atomic. We shall refer to the set of atoms of B_n as T_n; $T = \cup_{n>0} T_n$. We shall call algebras for which such measures exist *Constructive*.

Every finite algebra $\mathscr{B}$ is trivially constructive – put $B_0 = B_1 = B_2 \ldots = B$. In fact, we can characterise the constructive algebras exactly.

THEOREM 2.2. A Boolean Algebra is Constructive iff it is at most denumerably infinite.

Proof. If it is constructive it is obviously at most denumerably infinite. Conversely, if $\mathscr{B}$ is finite then it is constructive. If $\mathscr{B}$ is denumerably infinite choose an enumeration $\{b_n\}_{n>0}$ of $\mathscr{B}$ with 1 as the initial element – assume 0 absent from the enumeration. Let $B_0 = \{0, 1\}$. Let B_{n+1} be the algebra generated by $T_n \cup \{b_{n+1}\}$.

It will be helpful throughout this paper to think of the elements of B as sentences (equivalence classes) of some description language L. Let us assume, at least for the sake of argument, that there are possible worlds; let u be any such world and Σ_u be the set of elements $b \in B$ which represent sentences of L that are true in u. It follows that Σ_u is an ultrafilter of B.[8] On the other hand every ultrafilter Σ determines a class of possible worlds, where each u in the class satisfies $\Sigma = \Sigma_u$ – every consistent set of sentences determines at least one world in which every sentence in the set is true. Indeed we could, with Carnap, identify the set of possible worlds with the set of ultrafilters of B; but this identification has been criticised enough: it would render the notion of possible world intolerably linguistic in nature. But even if we reject this identification it remains true that the ultrafilters of $\mathscr{B}$ represent the totalities of information that can be expressed about the possible worlds in the language which B represents. Moreover, in formal contexts there is often no objection to using ultrafilters as 'pseudo-worlds'; and this is what I shall do here. In fact, to make explicit that they are only stand-ins for possible worlds I shall often refer to possible-worlds.

In the analysis of possible-worlds which follows we want to think of the partial descriptions b as clusters of possible-worlds – all those worlds in which b holds. It will be of some help, therefore, to write '$w \in b$' to stand for '$b \in w$' where b is an element of B and w an ultrafilter of $\mathscr{B}$. No confusion will be likely to arise from this inverted notation.[9]

In what follows let $U(\mathscr{B})$ denote the ultrafilters of $\mathscr{B}$.

It will be necessary in the sequel to be able to *approximate* to an ultrafilter w by ever more detailed partial descriptions. To this end we introduce the following concept.

DEFINITION 2.3. A sequence of elements from B, $\{s_n\}_{n\geqslant 0}$ where $s_n \in B_n$, is called an *ultrafilter sequence* (*uf*-sequence) iff for each n, s_n is an atom in B_n and $s_{n+1} \leqslant s_n$.

A set $D \subseteq B$ is *Generated* by a *uf*-sequence $\{s_n\}_{n\geqslant 0}$ exactly when $(\forall n)(D \cap B_n = \{b \in B_n : s_n \leqslant b\})$.

Our next result relates these *uf*-sequences to the ultrafilters of a constructive algebra. In fact, we prove that each ultrafilter u corresponds to a unique (up to the choice of measure) *uf*-sequence which generates it. The successive members s_n of this sequence may be regarded as representing ever more detailed descriptions of u, which *approximate* u in the sense that for any finite description b which is entailed by u there is an n such that b is entailed by s_n. We shall make constant use of the existence of these unique approximation sequences for possible-worlds.

THEOREM 2.4. Let $\mathscr{B}$ be a constructive algebra with measure d. Then the ultrafilters of $\mathscr{B}$ are exactly those sets generated by the ultrafilter sequences. Moreover, every such ultrafilter is generated by a unique ultrafilter sequence.

Proof. Let w be an ultrafilter in $\mathscr{B}$. Let $\hat{w}_n = B_n \cap w$ and $w_n = \cap \{b : b \in \hat{w}_n\}$ – this is defined as w_n is finite and $w_n \neq 0$ as w is an ultrafilter in $\mathscr{B}$. We claim that w_n is an atom in B_n. Suppose it is not; then there exists a $b \in B_n$, $b \neq 0$ and $b < w_n$. By definition $\sim (b \in \hat{w}_n)$ and so $\sim (b \in w)$. Since w is an ultrafilter $b^* \in w$ and as $b^* \in B_n$ we have $b^* \in B_n \cap w$ and so $b^* \geqslant w_n$ which contradicts $b < w_n$. Moreover any $b \in \hat{w}_n$ satisfies $w_n \leqslant b$ and so w_n 'generates' $\hat{w}_n$. It follows that w is generated by the sequence $\{w_n\}_{n\geqslant 0}$ – which is obviously unique.

Conversely, assume $w = \cup_{n=0}^{\infty} \hat{w}_n$ where $\hat{w}_n$ is generated by w_n and where $\{w_n\}_{n\geqslant 0}$ is a *uf*-sequence. Obviously $\hat{w}_n$ is an ultrafilter in $\mathscr{B}_n$; it follows that w is an ultrafilter in B.

We shall follow the convention used in this proof of denoting the *uf*-sequence of an ultrafilter for w by $\{w_n\}_{n>0}$.

3. COMPARATIVE PLAUSIBILITY

Our analysis of counterfactuals is to be based upon a relation of comparative plausibility on partial descriptions. To gain inspiration about the exact nature of this relation we are to be *guided by the principle* which informed our original discussion: a partial description a is at least as plausible as a description b just in case we can match refinements of b by refinements of a (of equal degree of detail), and which preserve the relation.

To aid our presentation of the axioms we introduce the notation

$$a \overset{w}{\geq} b \quad \text{where } a, b \in B, w \in U(B)$$

to mean that a is at *least as plausible* as b from the perspective of w.

Our first axiom just states that $\overset{w}{\geq}$ is a transitive relation.

(B1) $\quad a \overset{w}{\geq} b \;\&\; b \overset{w}{\geq} c \Rightarrow a \overset{w}{\geq} c.$

I take this axiom to be reasonably self evident: if a is at least as plausible as b and b is at least as plausible as c then a is at least as plausible as c.

Our second axiom is a principle of linearity: either a is at least as plausible as b or b is at least as plausible as a.

(B2) $\quad a \overset{w}{\geq} b \vee b \overset{w}{\geq} a.$

This condition stands or falls with the corresponding relation of comparative similarity on possible-worlds; a little reflection on our informal principle should convince the reader of this. If the relation of comparative similarity on possible-worlds is connected, and we make ever more detailed refinements of the partial descriptions, then the decision must be made one way or the other. This remark will eventually be shown to be a formal consequence of our theory. In any case, the addition of (B2) greatly simplifies the technical development of the theory – although this in itself would not be a sufficient reason for its inclusion. I tend to agree with D. Lewis that linearity is an essential part of our understanding of the notion of comparative similarity.

Principle (B3) is related to Lewis's condition of centering. If $b \in w$ and a, b are disjoint then b should be strictly more plausible than a (where b is strictly more plausible than a just in case it is false that a is at least as plausible as b).

(B3) $\quad (b \in w \;\&\; a \cap b = 0) \Rightarrow b \overset{w}{>} a$

where $b \overset{w}{\supseteq} a \Leftrightarrow_{\text{def}} \sim (a \overset{w}{\supseteq} b)$.

Once again this condition is strongly related to the corresponding principle for possible worlds (i.e., centering). I do, however, believe the principle (B3) to be more obvious than its analogue for possible worlds.

Our next two axioms are very closely related to our informal principle: a is at least as plausible as b exactly when we can match refinements of b by refinements of a, which are of equal degree of detail, and which preserve the relation of comparative plausibility.

(B4) $\quad a \overset{w}{\supseteq} b \;\&\; a' \geqslant a \Rightarrow a' \overset{w}{\supseteq} b$

(B5) $\quad a \overset{w}{\supseteq} b \;\&\; a \overset{w}{\supseteq} b' \Leftrightarrow a \overset{w}{\supseteq} b \cup b'$.

To observe this relationship we indicate that (B4) and (B5) are valid under the interpretation of $\supseteq$ induced by our informal principle. Suppose that $a \overset{w}{\supseteq} b$. Then since any refinement of a is a refinement of a' with $a' \geqslant a$, we have $a' \overset{w}{\supseteq} b$. Similarly, (B5) is valid under this interpretation: we need to observe only that a refinement of $b \cup b'$, *of sufficient degree of detail*, is a refinement of b or a refinement of b'; and conversely any refinement of b is a refinement of $b \cup b'$. We shall shortly make these arguments more precise.

Notice that the converse of (B4) actually follows from (B1)–(B5).

(B6) $\quad a \cup a' \overset{w}{\supseteq} b \Rightarrow a \overset{w}{\supseteq} b \vee a' \overset{w}{\supseteq} b$.

To see this suppose $a \cup a' \overset{w}{\supseteq} b$. By (B2) $a \overset{w}{\supseteq} a'$ or $a' \overset{w}{\supseteq} a$. Without loss of generality suppose the former. Since $a \overset{w}{\supseteq} a$, by (B5) $a \overset{w}{\supseteq} a \cup a'$. By (B1) we have $a \overset{w}{\supseteq} b$.

Our next axiom relates to the possibility that one of our partial descriptions describes no possible situations – or rather one of our partial descriptions contains contradictory information.

(B7) $\quad a \overset{w}{\supseteq} b \Rightarrow b \doteq 0 \vee \sim (a = 0)$

This seems very plausible; it states that if a is at least as plausible as b, and a is contradictory, then b must be contradictory also. Notice that, from (B7) (and (B2)), the following useful principle follows:

$$b = 0 \Rightarrow a \overset{w}{\supseteq} b$$

We shall make use of this later.

To make some of our informal remarks more precise we need to do the same for our informal principle. We have to exercise some care here because, presumably, the refinements we are interested in, when we claim that we can match any refinement of b by a corresponding refinement of the description a, have to be of a rather special type. They ought to be the 'best approximations' (to the possible-worlds) we can get, at the degree of detail specified. Otherwise, we have no guarantee that the refinement of a contains the same 'amount of information' as the refinement of b. Recall that T_k is the set of atoms of B_k. These atoms represent the best information available at the degree k. This brings us to a second modification of our principle. Surely, once we are in a position to *recognise* that $a \overset{w}{\geq} b$ will be verified within a finite time, then we may regard it as already verified. On the other hand, given our understanding of these relations of relative plausibility, if we are able to recognise that $a \overset{w}{\geq} b$ then surely we must have achieved this via some such verification process. With these remarks in mind we can state the formal version of our principle.

(B8) $$a \overset{w}{\geq} b$$
$$\Leftrightarrow$$
$$\{(b = 0) \vee (a \neq 0 \;\&\; (\exists k)(\forall m \geqslant k)(\forall b' \leqslant b, b' \in T_m)$$
$$(\exists a' \leqslant a, a' \in T_m)(a' \overset{w}{\geq} b'))\}.$$

The qualifications $b = 0, a \neq 0$, which occur, are to allow for the possibility of contradictory statements.

At last we are in a position to prove that (B8) is formally equivalent to our principles (B4), (B5) and (B7).

THEOREM 3.1. The Theories based upon axiom systems (B1), (B2), (B3), (B8) and (B1) through to (B7) are equivalent.

Proof. Assume (B1)–(B7). Assume $a \overset{w}{\geq} b$. We can assume by (B7) that $\sim (b = 0)$ and $\sim (a = 0)$. Let $k \geqslant d(a), d(b)$ and suppose $b' \leqslant b$ with $b' \in T_k$. By (B5) $a \overset{w}{\geq} b'$. Let $a_1, \ldots, a_m$ be the atoms in T_k such that $a = a_1 \cup \ldots \cup a_m$. By (B6) at least one of these a_i satisfies $a_i \overset{w}{\geq} b'$. Conversely, assume the right-hand side of (B8). Let $b_1, \ldots, b_m$ be the atoms of T_k (where k is larger than the guaranteed number for right-hand side to hold and also larger than $d(a)$ and $d(b)$) such that $b = b_1 \cup \ldots \cup b_m$. By hypothesis for *each* b_i, $1 \leqslant i \leqslant m$, there exists $a_i \in T_k$ such that $a_i' \overset{w}{\geq} b_i$. By (B4) for $1 \leqslant i \leqslant m$, a $\overset{w}{\geq} b_i$ and by (B5) $a \overset{w}{\geq} b$.

Assume (B1)–(B3) *and* (B8). (B7) follows immediately from (B8). We tackle (B4). If $b = 0$ the result is immediate. Assume $a \overset{w}{\supseteq} b$ and $b \neq 0$; since $b \neq 0$ by (B8) $a \neq 0$. Then by (B8) there exists a k such that for each $m \geqslant k$, $b' \leqslant b, b' \in T_m$ there exists $a' \in T_m, a' \leqslant a$ with $a' \overset{w}{\supseteq} b'$. But notice that $a' \leqslant a$ implies $a' \leqslant a \cup c$ for any c and so by (B8) $a \cup c \overset{w}{\supseteq} b$. For (B5) the cases where b or $b' = 0$ are trivial. First assume $a \overset{w}{\supseteq} b$ and $a \overset{w}{\supseteq} b'$. Let k be large enough to decide b and b' (i.e., $k \geqslant d(b), d(b')$) and bigger than the numbers guaranteed by the application of (B8) to $a \overset{w}{\supseteq} b$ and $a \overset{w}{\supseteq} b'$. Let $c \leqslant b \cup b'$ and $c \in T_k$. Then $c \leqslant b$ or $c \leqslant b'$. Without loss of generality assume the former. Then application of (B8) to $a \overset{w}{\supseteq} b$ guarantees an $a' \leqslant a, a' \in T_k$ with $a' \overset{w}{\supseteq} c$. Then (B8) gives $a \overset{w}{\supseteq} b \cup b'$. Conversely assume $a \overset{w}{\supseteq} b \cup b'$. Let $c \leqslant b$ with $c \in T_k$, where k is greater than the number guaranteed by application of (B8) to $a \overset{w}{\supseteq} b \cup b'$. Then since $c \leqslant b \cup b'$ there exists $a' \leqslant a, a' \in T_k$ with $a' \overset{w}{\supseteq} c$. It follows by (B8) that $a \overset{w}{\supseteq} b$. Similarly $a \overset{w}{\supseteq} b'$.

This result confirms our original intuitions; it further informs us that we might have taken the principle (B8) itself as fundamental. I cannot see that it matters much which we do except that (B4), (B5) make no mention of the measure function and so may be somewhat more acceptable if only for reasons of parsimony.

So far in this discussion of comparative plausibility we have paid little or no attention to the role of the actual world. In particular we might ask if the *recognition* that $\langle a, b \rangle$ belongs to the extension of $\overset{w}{\supseteq}$ can be achieved only on the basis of some finite partial description of w. One must address this question with some care for the answer will largely depend upon the notion of comparative plausibility in question. If one is interested in a more metaphysical notion of comparative plausibility, the answer to our question is probably no. But it should be clear by now that this is not our primary interest; we are more interested in a notion which will bring us closer to the discussion of counterfactuals given by Goodman. In which case it is unrealistic to insist that decisions of comparative plausibility are to be taken from the perspective of a complete description of the actual world.

To mitigate matters we begin by introducing the following definition:

$$a \overset{b}{\supseteq} c \leftrightarrows_{\mathbf{def}} (\forall w \supset b)(a \overset{w}{\supseteq} c).$$

Our beliefs about the role of the actual world can then be forced by the addition of a new axiom

(B9) $\quad a \overset{w}{\geq} b \Leftrightarrow (\exists n)(a \overset{w_n}{\geq} b).$

In words: if we are able to *recognise* that a is at least as plausible as b from the perspective of w, then we must be able to achieve this recognition on the basis of some finite partial description of w.

Identical considerations apply to the strict relations and lead to the following definition and axiom:

$$a \overset{b}{>} c \Leftrightarrow_{\mathbf{def}} (\forall w \ni b)(a \overset{w}{>} c)$$

(B10) $\quad a \overset{w}{>} c \Leftrightarrow (\exists n)(a \overset{w_n}{>} c).$

This may seem rather unsatisfactory, for principles (B9) and (B10) look rather circular; (B9), for example, can be rephrased in the following way:

$$a \overset{w}{\geq} b \Leftrightarrow (\exists n)(\forall v \ni w_n)(a \overset{v}{\geq} b).$$

This is not too serious an objection, however, for (B9) only states a regularity that any notion of comparative plausibility must obey. The intuition behind the principle is clear enough: we are able to recognise that a is at least as plausible as b, given w, exactly when there is a partial description of w of sufficient detail to decide the matter.

A *Boolean Frame* will be a constructive algebra B (together with a measure d) and a relation of relative plausibility which satisfies (B1)–(B10).

4. COMPARATIVE PLAUSIBILITY – TERNARY RELATIONS

There may, however, be some force to the objection of circularity in axioms (B9) and (B10). If anything it indicates that we should try to develop the theory directly in terms of a relation which makes no mention of possible-worlds.

Let the notation

$$a \overset{b}{\geq} c$$

indicate that a is at least as plausible as c given the information b (about the actual world). Of course, the information contained in any partial description b may not be sufficient to enable us to decide between $a \overset{b}{\geq} c$ and $c \overset{b}{\geq} a$; we cannot just define the strict relation $a \overset{b}{>} c$ in terms of the

negation of $a \overset{b}{\supseteq} c$. It seems necessary, therefore, to develop the theory in terms of two relations of relative plausibility. The relation $\supset$ is to consist of all triples $\langle a, b, c \rangle$ for which it is definite that a is strictly more plausible than c given b; while $\supseteq$ is to consist of those triples for which it is only definite that a is at least as plausible as c given b. We collect together the properties of these relations in the following axiom system.

AXIOMS. *For* $\supseteq, \supset$. For $a, a', b, b', c, c' \in B$ and in (P1)–(P12), $b, b' \neq 0$.

(P1) $\quad a \overset{b}{\supseteq} c \,\&\, c \overset{b}{\supseteq} d \Rightarrow a \overset{b}{\supseteq} d$

(P2) $\quad a \overset{b}{\supseteq} c \,\&\, a \overset{b'}{\supseteq} c \Leftrightarrow a \overset{b \cup b'}{\supseteq} c$

(P3) $\quad a \overset{b}{\supseteq} c \,\&\, a' \geqslant a \Rightarrow a' \overset{b}{\supseteq} c$

(P4) $\quad a \overset{b}{\supseteq} c \,\&\, a \overset{b}{\supseteq} c' \Leftrightarrow a \overset{b}{\supseteq} c \cup c'$

(P5) $\quad a \overset{b}{\supset} c \,\&\, c \overset{b}{\supset} d \Rightarrow a \overset{b}{\supset} d$

(P6) $\quad a \overset{b}{\supset} c \,\&\, a \overset{b'}{\supset} c \Leftrightarrow a \overset{b \cup b'}{\supset} c$

(P7) $\quad a \overset{b}{\supset} c \,\&\, a' \geqslant a \Rightarrow a' \overset{b}{\supset} c$

(P8) $\quad a \overset{b}{\supset} c \,\&\, a \overset{b}{\supset} c' \Leftrightarrow a \overset{b}{\supset} c \cup c'$

(P9) $\quad a \cap b = 0 \Rightarrow b \overset{b}{\supset} a$

(P10) $\quad (\exists k)(\forall b \in T_k)(a \overset{b}{\supset} c \vee c \overset{b}{\supset} a \vee a \overset{b}{\equiv} c)$

(P11) $\quad a \overset{b}{\supset} c \Rightarrow a \overset{b}{\supseteq} c \,\&\, \sim (c \overset{b}{\supseteq} a)$

(P12) $\quad c = 0 \,\&\, a \neq 0 \Rightarrow a \overset{b}{\supset} c$

(P13) $\quad a \overset{0}{\supseteq} c \,\&\, a \overset{0}{\supset} c$

Here in (P10) $a \overset{b}{\equiv} c \Leftrightarrow_{\text{def}} (a \overset{b}{\supseteq} c \,\&\, c \overset{b}{\supseteq} a)$.

Most of the axioms are direct analogues of the axioms of $\overset{w}{\supseteq}$: (P1), (P5) correspond to (B1); (P3), (P7) correspond to (B4); (P4), (P8) to (B5) and (P9) to (B3). Axiom (P2) (and (P6)) incorporates two basic insights. If we are in a position to *recognise* that $a \overset{b}{\supseteq} c$, then we must be in a position to recognise $a \overset{b'}{\supseteq} c$ for all extensions b' of b. This is the principle incorporated in (P2) from right to left. The converse embodies the idea that if we are in a position to recognise that the relation will be verified within a finite time, then we may regard it as already verified. In fact, if $(\exists k)(\forall b' \leqslant b)(d(b') \geqslant k)(a \overset{b'}{\supseteq} c)$, then using (P2) we can establish that $a \overset{b}{\supseteq} c$. Principle (P10) corresponds to (B2). Because $\supset$, $\supseteq$ are not interdefinable we cannot simplify this axiom to

$$(\exists k)(\forall b \in T_k)(a \overset{b}{\supseteq} c \vee c \overset{b}{\supseteq} a).$$

The only relationship between $\supset$, $\supseteq$ is given by (P11) and this is not sufficient to guarantee the equivalence between the above and (P10). One more observation about (P10) seems to be in order. Implicitly, it makes reference to possible worlds through the sets T_k. There seems no obvious way of removing this reference; we need to quantify with respect to finite sets from the algebra. Axiom (B7) corresponds to (P12). Axiom (P12) may appear quite strange; it embodies the principle that from a contradiction anything follows.

As an afterthought, one might enquire whether or not something like the following principle is valid:

$$a \overset{b}{\supseteq} c \Leftrightarrow (\exists k)(\forall m \geqslant k)(\forall c' \leqslant c, c' \in T_m)(\exists a' \leqslant a, a' \in T_m)(a' \overset{b}{\supseteq} c').$$

After all this is the analogue of our original guiding principle. It is at least as strong as our original principle; one direction is straightforward and the other follows from an easy application of König's lemma. Rather than discuss the proof of this remark I wish to consider the principle itself. I believe it to be intuitively unsound. I have no objection to the inference from right to left; this much follows from our theory and in particular (P3) and (P4). It is the *converse* I find controversial. Suppose for argument's sake that $a = b$. The reason why we are prepared to assert $a \overset{a}{\supseteq} c$ is clear: no matter how we extend c (to c' say) and a (to a' say) we can find an extension of a (a' *itself*) which preserves the relation. But the converse claims more. According to it we must be able to find, for each extension c' of c, an extension a' of a which will maintain the relation for all extensions a'' of a. This follows from (P2). This seems intuitively unsound.

The next few results prove that the two theories are in complete agreement. The one based totally on partial descriptions ((P1)–(P13)) seems the more satisfactory. If our original claim that we can never command more than a partial description of any possible world is correct, then presumably this applies to the actual world. The awkwardness of this theory counts against it in that we require a more elaborate axiom system. For technical convenience, therefore, I shall mostly use the system based upon (B1)–(B10).

THEOREM 4.1. Let $\overset{w}{\supseteq}$ satisfy (B1)–(B10). Define,

$$a \overset{b}{\supseteq} c \Leftrightarrow (\forall w \ni b)(a \overset{w}{\supseteq} c)$$
$$a \overset{b}{\supset} c \Leftrightarrow (\forall w \ni b)(a \overset{w}{\supset} c);$$

then these relations satisfy axioms (P1)–(P13).

Proof. (P1) and (P5) follows from (B1); (P3) from (B4) and (P7) from (B5); (P4) follows from (B5) and (P8) from (B6) and (B4). For axiom (P9) assume $a \cap b = 0$; then we need to show $a \overset{a}{\supset} b$, i.e., $(\forall w \ni a)(a \overset{w}{\supset} b)$. But this follows immediately from principle (B3). Let's turn our attention to (P2) ((P6)). This follows immediately from the observation that for every w, $w \ni b \cup b'$ iff $w \ni b$ or $w \ni b'$. For (P10) we begin with (B2) for each w, $a \overset{w}{\supseteq} b \vee b \overset{w}{\supseteq} a$. Hence $\forall w(a \overset{w}{\supset} b \vee b \overset{w}{\supset} a \vee b \overset{w}{\equiv} a)$. By (B9) and (B10) we have $(\forall w)(\exists n)(a \overset{w_n}{\supset} b \vee b \overset{w_n}{\supset} a \vee b \overset{w_n}{\equiv} a)$. Hence, by König's lemma $(\exists n)(\forall w)(a \overset{w_n}{\supset} b \vee b \overset{w_n}{\supset} a \vee b \overset{w_n}{\equiv} a)$. Axiom (P11) follows by the definitions and the observation that $a \overset{b}{\supset} c$ implies $(\forall w \ni b)(a \overset{w}{\supseteq} c)$ & $(\forall w \ni b)(\sim (c \overset{w}{\supseteq} a))$. (P12) follows directly from (B7). (P13) follows by definition.

THEOREM 4.2. Let $\overset{w}{\supseteq}$ be the relation derived from a theory based on axioms (P1)–(P13), and which is defined by

$$a \overset{w}{\supseteq} b \Leftrightarrow (\exists n)(a \overset{w_n}{\supseteq} b).$$

Then $\overset{w}{\supseteq}$ satisfies (B1)–(B10).

Proof. Once again much, if not all, is tedious but straightforward. (B1) follows from (P1) & (P2); (B4) from (P3), and (B5) from (P4) & (P2). For (B2) we need to show $(\exists n)(a \overset{w_n}{\supseteq} b) \vee (\exists n)(b \overset{w_n}{\supseteq} a)$. But (P10) implies $(\exists n)(\forall w)(a \overset{w_n}{\supseteq} b \vee b \overset{w_n}{\supseteq} a)$. Hence $(\forall w)(\exists n(a \overset{w_n}{\supseteq} b) \vee \exists n(b \overset{w_n}{\supseteq} n))$. So (B2) holds. Assume $a \cap b = 0$ and $w \ni a$. We know from (P9) that $a \overset{a}{\supset} b$. Hence, since $(\exists k)(w_k \leqslant a)$, we have $(\exists k)(a \overset{w_k}{\supset} b)$ by (P6). Hence (B3) is true. We now need to check (B9) and (B10). Assume $a \overset{w}{\supset} b$. Then by definition we have $\sim (b \overset{w}{\supseteq} a)$, i.e. $(\forall k)(\sim (b \overset{w_k}{\supseteq} a))$. By (P10) $(\exists k)(a \overset{w_k}{\supset} b \vee b \overset{w_k}{\supset} a \vee b \overset{w_k}{\equiv} a)$. The second and third possibilities are ruled out by (P2) and the fact that $(\forall k)(\sim (b \overset{w_k}{\supseteq} a))$. Hence $(\exists k)(a \overset{w_k}{\supset} b)$. Conversely, assume $(\exists k)(a \overset{w_k}{\supset} b)$. Suppose $b \overset{w}{\supseteq} a$. Then by definition $(\exists k)(b \overset{w_k}{\supseteq} a)$. Hence, by (P2) $(\exists k)(\forall m \geqslant k)(b \overset{w_m}{\supseteq} a)$. So for k large enough we have (this time by (P6)) $(\forall m \geqslant k)(b \overset{w_m}{\supseteq} a \,\&\, a \overset{w_m}{\supset} b)$ and this cannot be (by P11). Thus (B10) holds – (B9) is just a definition. (B7) follows from (P12).

As a bonus, the way the three place relations are defined in the original theory, is a formal consequence of (P1) through (P13).

THEOREM 4.3. For a, b, c in B we have:

(1) $\quad a \overset{b}{\geq} c \Leftrightarrow (\forall w \ni b)(a \overset{w}{\geq} c)$

(2) $\quad a \overset{b}{>} c \Leftrightarrow (\forall w \ni b)(a \overset{w}{>} c)$

where $\overset{w}{\geq}$ is derived relation of the theory (P1)–(P13).

Proof. Assume $a \overset{b}{\geq} c$. Then the right-hand side follows from (P2) and the definition $a \overset{w}{\geq} c \Leftrightarrow (\exists n)(a \overset{w_n}{\geq} c)$. Conversely, assume the right-hand side. By definition and König's lemma $(\exists k)(\forall w \ni b)(a \overset{w_k}{\geq} c)$. (P2) gives immediately that $a \overset{b}{\geq} c$. For $b = 0$ use (P13). Part (2) is proven similarly.

5. TRUTH CONDITIONS FOR COUNTERFACTUAL SENTENCES

We shall provide the truth conditions for a simple propositional language containing counterfactual operators. Let L be any countable language, then $L_{\Box\!\!\rightarrow}$ is the smallest superset of L that is closed under the truth functional connectives $\sim$, & and the counterfactual operator $\Box\!\!\rightarrow$. The other logical connectives $\vee, \rightarrow, \Leftrightarrow$ and the intensional $\Diamond\!\!\rightarrow$ (if – then it might be the case that –) are given by definition in the normal way.

Our truth conditions are given in a rather unorthodox way. Briefly, we define two semantic relations $b \models P$ and $b \dashv P$ by (simultaneous) induction on P. Intuitively, $b \models P$ expresses the fact that b contains sufficient information to assert P; and $b \dashv P$ expresses the fact that b contains enough information to refute P. There is, of course, no reason to expect these relations to be interdefinable. We may not have sufficient information to decide the matter one way or the other so, in particular, we cannot infer that $b \dashv P$ holds from the fact that $b \models P$ does not hold; we cannot define $b \dashv P$ as $\sim (b \models P)$.

We are going to give the truth-conditions for $L_{\Box\!\!\rightarrow}$ in terms of some Boolean frame B. You will recall that our algebras can be thought of as induced by some underlying description language DL. What exactly is the relationship between L, $L_{\Box\!\!\rightarrow}$ and DL? Presumably $L \subseteq DL$. A *Boolean model* $M = \langle B, f \rangle$ consists of a Boolean frame B together with a

function $f: L \to B$. This function might be seen as just reflecting this inclusion. We might, however, demand more than this, namely that L (or $L_{\Box\to}$) and DL actually coincide. This is much more problematic. To argue for this view is to take what might be described as a 'Nominalist position'. We are demanding that any distinctions that we are able to draw between different possible situations can be articulated in the object language. But surely this is not the only possible and perhaps not even the most plausible view. We might well have the capacity to draw finer situational distinctions than those expressible in the object language.

We provide the truth-conditionals for $L_{\Box\to}$ gradually. Initially we consider simple counterfactuals, namely those whose antecedents and consequents are *not* themselves counterfactuals. Let L' be the smallest superset of L which is closed under the truth-functional connectives and let $L_{\Box\to}(1)$ be the smallest superset of L' which contains $P_{\Box\to}Q$ for P, Q in L'. The reason for this is entirely pedagogical. In order to provide the truth-conditions for more complex counterfactuals we need to associate with each simple counterfactual an element of the algebra. The fact that we are able to do this will follow from our truth-conditions for simple counterfactual conditions.

Let $M = \langle B, f \rangle$ be a Boolean model. We define $b \models P$ and $b \dashv\models P$ where b is in B and P is in $L_{\Box\to}(1)$. The definition is by (simultaneous) induction on P.

(i) $b \models P$ iff $b \leqslant f(P)$; P in L
$b \dashv\models P$ iff $b \leqslant f(P)^*$; P in L.

(ii) $b \models P \,\&\, Q$ iff $b \models P$ and $b \models Q$
$b \dashv\models P \,\&\, Q$ iff $b \dashv\models P$ or $b \dashv\models Q$.

(iii) $b \models \sim P$ iff $b \dashv\models P$
$b \dashv\models \sim P$ iff $b \models P$.

This completes the truth-conditions for L'. Before we give the clause for the counterfactual we indicate how to extend the function f to L':

$$f(P \,\&\, Q) = f(P) \cap f(Q)$$
$$f(\sim P) = f(P)^*.$$

LEMMA 5.1. For each b in B and P in L' the following holds.

(1) $b \models P$ and $b' \leqslant b$ implies $b' \models P$

(2) $b \dashv\models P$ and $b' \leqslant b$ implies $b' \dashv\models P$.

The result follows by a simple inductive argument. The next result links the function f with the relations $\models$, $\dashv\models$.

LEMMA 5.2. Let $P \in L'$. Then there exists an n such that for all $m \geqslant n$, for each $b \in T_m$

(1) $b \models P$ iff $b \leqslant f(P)$

(2) $b \dashv\models P$ iff $b \leqslant f(P)^*$.

Proof. First define $d: L' \to \omega$ by recursion as follows:

(1) $d(P) = d(f(P))$ for $P \in L$

(2) $d(\sim P) = d(P)$

(3) $d(P \,\&\, Q) = \max(d(P), d(Q))$

Notice that $d(P) \geqslant d(f(P))$. The number $d(P)$ is the number we want; we prove this by induction on P.

The case $P \in L$ is trivial as indeed is the case for negation. We concentrate on the conjunction. Suppose $b \models Q \,\&\, R$. This is equivalent to $b \models Q$ and $b \models R$. Choose b such that $b \in T_m$, $m \geqslant d(Q \,\&\, R)$. Then $b \models Q$ and $b \models R$ is equivalent to $b \leqslant f(Q)$ and $b \leqslant f(R)$ – by induction hypothesis. Next suppose $b \dashv\models Q \,\&\, R$. This is equivalent to $b \dashv\models Q$ or $b \dashv\models R$. Once again let $b \in T_m$, $m \geqslant d(Q \,\&\, R)$. This is then equivalent to $b \leqslant f(Q)^*$ or $b \leqslant f(R)^*$. But b is an atom in T_m where $m \geqslant d(Q \,\&\, R) \geqslant d(f(Q)), d(f(R))$; hence $b \leqslant f(Q)^*$ or $b \leqslant f(R)^*$ is equivalent to $b \leqslant f(Q)^* \cup f(R)^*$.

We shall refer to $d(P)$ as the *Degree of P*.

THEOREM 5.3. For each P in L' we have: $(\exists k)(\forall t \in T_k)(t \models P \vee t \dashv\models P)$.

Proof. Put $k = d(P)$. Since $k \geqslant d(f(P))$ we know that for any $t \in T_k$, $t \leqslant f(P)$ or $t \leqslant f(P)^*$. By the previous theorem $t \models P$ or $t \dashv\models P$.

To provide the truth conditions for the counterfactual itself our strategy will be to give an account of our informal procedure for the assessment of counterfactuals. If you recall it was of the following form. First we

must think of a number of alternative circumstances in which the antecedent could be (would be) true. Next we select the most plausible branches (with respect to our relations of relative plausibility). We then check to see if these most plausible alternatives, together with the antecedent, imply the consequent; otherwise we partition the antecedent situations more finely. Let M be a Boolean model and let $P \in L'$. A *P-decision* tree p, is any finite tree p, labelled with elements from B with $f(P)$ at the vertex, and which satisfies the following conditions.

(i) For each branch t of p, $\hat{t} \neq 0$ ($\hat{t}$ is the intersection of the elements of t – where it is clear what is meant we shall often just write t rather than $\hat{t}$).

(ii) If a is a node of p with immediate ancestors $a_1, \ldots, a_m$ then $a_1 \cup \ldots \cup a_m = a$ and $a_i \cap a_j = 0, i \neq j$.

Let $b \in B$. We shall call p a $\langle b, P \rangle$-*decision* tree if p is a P-decision tree which satisfies condition (iii) below. Where P is unimportant we shall refer to a b-decision tree.

(iii) Let $B(p)$ denote the branches of p. Then $s, t \in B(p)$ implies

$$s \overset{b}{\supset} t \vee t \overset{b}{\supset} s \vee t \overset{b}{\equiv} s.$$

The set of "most plausible" branches of p, with respect to b, is the set $N_p(b) = \{s \in B(p)\colon \forall t \in B(p), s \overset{b}{\supseteq} t\}$.

The notion of $a\langle b, P\rangle$-decision tree is meant to be a precise counterpart of the informal account; the various layers in the decsion tree are meant to reflect the finer and finer partitions of the antecedent. I take it that when we refine the antecedent we wish to do this in a consistent way: we induce the finer partitions by the addition of information which is neither contradicted nor entailed by that already accumulated. This is the content of condition (i). The second condition is included to ensure that our refinements are genuine partitions of the antecedent, i.e., we do not lose any antecedent situations. Condition (iii) is somewhat less obvious; it is included to ensure that b contains sufficient information to decide the most plausible branches. We shall shortly say a little more about this condition.

This brings us to the clause in the truth conditions for the counterfactual conditional. The idea should be obvious by now: a counterfactual is true just in case there is a decision tree which guarantees it to be.

(iv) $b \models P \mathbin{\Box\!\!\rightarrow} Q$ iff there exists a $\langle b, P \rangle$-decision tree p, where each branch c decides Q and where for *each* branch c in $N_p(b)$, $\hat{c} \models Q$

$b \dashv P \mathbin{\Box\!\!\rightarrow} Q$ iff there exists a $\langle b, P \rangle$-decision tree p, where each branch c decides Q and for *some* branch c in $N_p(b)$, $\hat{c} \dashv Q$

where in (iv) a branch c decides Q if and only if $\hat{c} \models Q$ or $\hat{c} \dashv Q$.

The truth-conditions for $P \mathbin{\Diamond\!\!\rightarrow} Q$, the "might" conditional, can be deduced from the definition $P \mathbin{\Diamond\!\!\rightarrow} Q \Leftrightarrow_{\text{def}} \sim (P \mathbin{\Box\!\!\rightarrow} \sim Q)$.

THEOREM 5.4. Let B be a Boolean frame. Then for any $b \neq 0$, $b' \neq 0$ in B with $b' \leqslant b$ we have:

(1) $b \models P \mathbin{\Box\!\!\rightarrow} Q$ implies $b' \models P \mathbin{\Box\!\!\rightarrow} Q$

(2) $b \dashv P \mathbin{\Box\!\!\rightarrow} Q$ implies $b' \dashv P \mathbin{\Box\!\!\rightarrow} Q$.

Proof. Let p be a $\langle b, P \rangle$-decision tree guaranteeing that $b \models P \mathbin{\Box\!\!\rightarrow} Q$. Then (since if $b' \leqslant b$ and $a \overset{b}{\supseteq} c$ then $a \overset{b'}{\supseteq} c$) p is also a $\langle b', P \rangle$-decision tree, and $N_p(b) \subseteq N_p(b')$. On the other hand, suppose $s \in N_p(b')$ and $s \notin N_p(b)$. Then by the following lemma there is some $t \in B(p)$ such that $t \overset{b}{\supset} s$. By (P6) $t \overset{b'}{\supset} s$ and so we have a contradiction by P11. Hence $N_p(b) = N_p(b')$. (1) and (2) now follow.

LEMMA 5.5. For each decision tree p

$$N_p(b) = \cap \{A: 0 \neq A \subseteq B(p) \,\&\, x \in B(p) - A \,\&\, y \in A \Rightarrow y \overset{b}{\supset} x\}$$

Proof. Let $N = \{A: 0 \neq A \subseteq B(p) \,\&\, x \in B(p) - A \,\&\, y \in A \Rightarrow y \overset{b}{\supset} x\}$. First claim $N_p(b) \in N$. Now $N_p(b) \subseteq B(p)$, and if $\sim (x \in N_p(b))$ and $y \in N_p(b)$ then $y \overset{b}{\supseteq} x$ and $\sim (x \overset{b}{\supseteq} y)$. By condition (iii) in our definition of decision trees $y \overset{b}{\supset} x$. Hence $N_p(b) \in N$. Hence $\cap N \subseteq N_p(b)$. Conversely, if $z \in N_p(b)$ then if $A \in N$ and $z \notin A$ we know that $y \overset{b}{\supset} z$ for each $y \in A$. But $z \overset{b}{\supseteq} y$ as $z \in N_p(b)$ – contradiction. Hence $N = 0$. But this is impossible, since $N_p(b) \in N$.

This last theorem is in keeping with our intuitive understanding of the

relations $\models$ and $\dashv$: the relations should be preserved under possible increases in information. Once a decision concerning a counterfactual has been taken further refinements will not alter matters. But how can we be sure that the truth value of a counterfactual is independent of the particular way that we partition the antecedent? The following result should relieve any anxiety over this matter.

THEOREM 5.6. For any Boolean frame B, with $b \neq 0$ in B, we have:

(1) $b \models P \Box\!\!\rightarrow Q$ iff for each $\langle b, P\rangle$-decision tree p, (where each branch decides Q), *each* $c \in N_p(b)$ satisfies $c \models Q$.

(2) $b \dashv P \Box\!\!\rightarrow Q$ iff for each $\langle b, P\rangle$-decision tree p, (where each branch decides Q) *some* $c \in N_p(b)$ satisfies $c \dashv Q$.

Proof. A little notation will facilitate the presentation of the proof. Suppose $A \subseteq B(p) - A$ is some set of branches in p. For example, $A = \{a_1, \ldots, a_n\}$. Let $\hat{a}_i$ be the last element on the branch a_i and let $\overline{A} = \hat{a}_1 \cup \ldots \cup \hat{a}_n$. Suppose contrary to the theorem $b \models P \Box\!\!\rightarrow Q$ and for some $\langle b, P\rangle$-decision tree q (where each b branch decides Q) some $c_q \in N_q(b)$ satisfies $\sim (c_q \models Q)$ (and, therefore, $c_q \dashv Q$ as each branch decides Q). We shall deduce a contradiction.

Let p be the decision tree which guarantees the truth of $b \models P \Box\!\!\rightarrow Q$. First notice

(i) $\overline{N_p(b)} \overset{b}{\supseteq} \overline{B(p) - N_p(b)}$ and

(ii) $\hat{c}_q \overset{b}{\supseteq} \overline{B(q) - \{c_q\}}$.

By hypothesis, $\hat{a} \models Q$ for each $a \in N_p(b)$. Now either some such $\hat{a}$ satisfies $\hat{a} \cap \hat{c}_q \neq 0$ or $\hat{c}_q \cap \overline{N_p(b)} = 0$. If there is such an $a \in N_p(b)$ such that $\hat{a} \cap \hat{c}_q \neq 0$ then $\hat{a} \cap \hat{c}_q \models Q \,\&\, \sim Q$. But this is a contradiction (since a simple inductive argument shows that if $b \models Q \,\&\, \sim Q$ then $b = 0$ in B). Hence $\hat{c}_q \cap \overline{N_p(b)} = 0$. So $\hat{c}_q \leqslant \overline{B(p) - N_p(b)}$ – as $\hat{c}_q \leqslant f(P)$ and $f(P) = \overline{B(p)}$. Similarly, $\overline{N_p(b)} \leqslant \overline{B(q) - \{c_q\}}$. So, by (i) and (P8), $\overline{N_p(b)} \overset{b}{\supseteq} \hat{c}_q$ and, by (ii) and (P4), $\hat{c}_q \overset{b}{\supseteq} \overline{N_p(b)}$. This is a contradiction by (P11). The proof of the remaining half of the theorem is similar.

THEOREM 5.7. For any Boolean frame B with $b \in B$, if $b \models P \Box\!\!\rightarrow Q$ and $b \dashv\models P \Box\!\!\rightarrow Q$ then $b = 0$.

Proof. By the theorem if $b \neq 0$ then any decision tree p for P must have some $c \in N_p(b)$ such that $\hat{c} \models P$ and $\hat{c} \dashv\models P$. This cannot be (induction of $P \in L'$) unless $c = 0$. But then, by the condition (i) in the definition decision trees, this is impossible. Hence $b = 0$.

The reader can perhaps now see the necessity for condition (iii) in our definition of decision trees: it ensures that a decision concerning the truth or falsity of a counterfactual does not need to be withdrawn in the light of new information.

We can connect our account with that of Goodman. On his view (roughly) $P \Box\!\!\rightarrow Q$ is backed by an argument or derivation $P, P_1, \ldots, P_n \vdash Q$, where P_i are 'cotenable' with P and where there is *no* such derivation for $\sim Q$. Our notion of comparative plausibility is obviously closely allied to Goodman's notion of cotenability; indeed if we translate Goodman's criteria into the terms of comparative plausibility we assert P is 'cotenable' with Q just in case $P \,\&\, Q$ is at least as plausible as $P \,\&\, \sim Q$. If this is a correct interpretation of Goodman's notion of cotenability then our truth conditions provide the appropriate derivation of the consequent from the antecedent: the most plausible branches are cotenable with the antecedent and imply the consequent. The previous theorem provides the second half of Goodman's criteria in that it ensures that no such derivation for the negation of the consequent can exist. This is, of course, not a total solution to Goodman's problem, but it is the best we can do in such a general framework.

Our discussion highlights a problem which has played a central part in the theorising about counterfactuals; and not just about counterfactuals but about conditionals in general; indeed it crops up at almost every point in the philosophy of language. The problem concerns the relationship between the truth-conditions of sentences and the means that we have available for verifying them. Do the evaluation procedures we have just described capture the truth-conditions of counterfactuals exhaustively, or should they rather be seen as modelling the accessible means of assessing truth?

Questions of this sort are notoriously difficult. It is, of course, always dangerous to claim that no more of interest can be said about a certain question but nevertheless some such questions do seem unanswerable.

Indeed, the insight that there really is no reason to expect such questions to be conclusively answered is itself a valuable change of attitude in language theory; just as it was a helpful change of attitude when people gave up the idea that a semantic analysis would be inadequate unless it attributes to the sentence analysed a truth-value in all circumstances or contexts of use. The reason why we should not automatically expect there to be answers to such questions is this: as long as there is a fair measure of agreement in the means of verification (for proposition of a certain type) that users have in common, so that they come in the majority of cases to the same conclusion regarding the truth-value of such a proposition when they apply the means of evaluation, that each has at his disposal, the sentences in question will serve adequately as vehicles of communication. The question whether the methods of evaluation used do themselves constitute the truth-conditions for these sentences need never be settled; there may be a gap in truth-conditions here just as truth-gaps arise in so many areas of language. It is always possible to *adopt* truth-conditions which transcend the procedures of evaluation that we recognise as tied to the sentence type in its actual use. This is perfectly acceptable as long as these new truth-conditions do not significantly conflict with the procedures in all those cases where the latter yield a definite result; as long, that is, as the imposition of truth-conditions respect the instances where truth-conditions are decidable. Indeed, in many instances, there may be valid theoretical or technical reasons for such impositions. We should, however, be fully aware of what it is that we are doing. We are reforming old (or even introducing new) vernacular and not merely spelling out what was already there to be discovered.

On the face of it our relations $\models$, $\dashv$ admit truth-value gaps, for as we have previously stated it may not be the case that either $b \models P$ or $b \dashv P$ for some particular description b. We may not know enough to decide the matter. Nevertheless, we may reach some point where for a particular P we may be able to choose one way or the other. In other words, for a description b, *of sufficient degree of detail*, either b entails P or b entails $\sim P$.

THEOREM 5.8. For any model M if $f(P) \neq 0$ then

$$(\exists k)(\forall b \in T_k)(b \models P \,\Box\!\!\rightarrow Q \vee b \dashv P \,\Box\!\!\rightarrow Q).$$

Proof. Consider the decision tree p with $f(P)$ at the vertex and its

ancestors those atoms t of B_n ($n \geqslant d(Q), d(P)$) such that $t \leqslant f(P)$. By (P10) we can choose k large enough to 'decide' modulo $\supseteq, \supset$ the branches of p, i.e., for each pair of branches of p, x, y say, and for each b in T_k

$$x \overset{b}{\supseteq} y \vee y \overset{b}{\supseteq} x \vee x \overset{b}{\equiv} y.$$

But then for each $b \in T_k$ and each $x \in N_p(b)$ either $x \models Q$ or $x \dashv Q$. Hence, $b \models P \Box\!\!\rightarrow Q$ or $b \dashv P \Box\!\!\rightarrow Q$.

We shall refer to the k uncovered in this result as the *degree* of $P \Box\!\!\rightarrow Q$ and write $d(P \Box\!\!\rightarrow Q) = k$.

Despite this result, our theory of counterfactuals may well appear as a square refusal to accept such transcendental truth-conditions. If anything, the very fact that, according to our theory, any counterfactual can apparently be decided within a finite amount of time strengthens this view. This is to be seen in contrast to the semantic analysis offered by Lewis and Stalnaker. This, however, is not the only possible interpretation of our theory.

The truth of a counterfactual is said, by the present theory, to consist in the existence of a certain decision tree. Unlike possible worlds the relevant portion of the tree can be known completely by the language user; and this is, of course, part of the motivation behind the present theory. Despite this fact, the statement that the partition tree has the required property may well transcend what is directly verifiable. The main problem concerns the consistency checks involved in the construction of the decision tree. So far it has been implicitly assumed that these verifications of consistency are instantaneous. A much more realistic approach would require these verifications to be carried out as part of the partition process itself. The important observation here concerns the termination of such a verification process: if *DL* contains predicate logic then such a process cannot, in general, be completed within a finite number of steps.

At long last we can provide the promised extension of the function f to the whole of $L_{\Box\!\!\rightarrow}(1)$.

THEOREM 5.9. Let M be some Boolean model. Then f can be extended to a function $f: L_{\Box\!\!\rightarrow}(1) \rightarrow B$ such that for each $b \in T_k$, $k \geqslant d(P \Box\!\!\rightarrow Q)$, if $f(P) \neq 0$, then

(1) $\qquad b \leqslant f(P \Box\!\!\rightarrow Q) \quad$ iff $b \models P \Box\!\!\rightarrow Q$

(2) $\quad b \leqslant f(P \Box\!\!\rightarrow Q)^* \quad$ iff $b \dashv\models P \Box\!\!\rightarrow Q$.

(If $f(P) = 0$ put $f(P \Box\!\!\rightarrow Q) = 1$.)

Proof. Let $k \geqslant d(P \Box\!\!\rightarrow Q)$. Let $b_1, \ldots, b_m$ be the atoms of B_k. Let $b_1, \ldots, b_n$ be the atoms such that $b_i \models P \Box\!\!\rightarrow Q$, $1 \leqslant i \leqslant n$ and $b_{n+1}, \ldots, b_m$ the atoms such that $b_j \dashv\models P \Box\!\!\rightarrow Q$, $n+1 \leqslant j \leqslant m$. Let $f(P \Box\!\!\rightarrow Q) = b_1 \cup \ldots \cup b_n$. If there are no atoms such that $b_i \models P \Box\!\!\rightarrow Q$ then of course $f(P \Box\!\!\rightarrow Q)$ is put equal to 0.

This result is the key to the treatment of more complex counterfactuals – those which have antecedents and/or consequents which are themselves counterfactuals. Our definition of a model provided a function $f: L \to B$; we were then able to give a truth theory for sentences in $L_{\Box\!\!\rightarrow}$ which involves only *one* level of conterfactual conditionals. Our theorem guarantees the existence of a function $\hat{f}: L_{\Box\!\!\rightarrow}(1) \to B$. In its turn this enables us to provide truth conditions for sentences in $L_{\Box\!\!\rightarrow}$ which involve two levels of counterfactuals – $L_{\Box\!\!\rightarrow}(2)$. So far we have said little about the problem of counterfactuals in $L_{\Box\!\!\rightarrow}(1)$ which have impossible antecedents (in terms of the model $f(P) = 0$ where P is the antecedent of the conditional). Lewis argues for the truth of such counterfactuals, and to facilitate comparison with his theory; we have followed suit by putting $f(P \Box\!\!\rightarrow Q) = 1$ for $f(P) = 0$. Proceeding in this way we can define a sequence of functions $\{f_n\}_{n \geqslant 0}$ where

$$f_0 = f, \qquad f_{n+1} = \hat{f}_n$$

where $f_n: L_{\Box\!\!\rightarrow}(n) \to B$, $L_{\Box\!\!\rightarrow}(0) = L$ and $L_{\Box\!\!\rightarrow}(n+1)$ is obtained from $L_{\Box\!\!\rightarrow}(n)$ as follows. First close $L_{\Box\!\!\rightarrow}(n)$ under the truth-functional operators $(\&, \sim)$ to obtain $L'_{\Box\!\!\rightarrow}(n)$. $L_{\Box\!\!\rightarrow}(n+1)$ is the smallest superset of $L'_{\Box\!\!\rightarrow}(n)$ containing $P \Box\!\!\rightarrow Q$ for $P, Q \in L'_{\Box\!\!\rightarrow}(n)$. Let $L_{\Box\!\!\rightarrow} = \cup_{n=0} L_{\Box\!\!\rightarrow}(n)$. We can define

$$f_\infty: L_{\Box\!\!\rightarrow} \to B$$

by $f_\infty = \cup_{n=0} f_n$ where in the obvious sense $f_n \subseteq f_{n+1}$ (f_{n+1} is an extension of f_n). We shall write f for f_∞ to avoid cumbersome notation.

Our last result is bound to provoke certain misgivings. It ascertains that each counterfactual sentence P "coincides" with an element b_P of the Boolean frame. But in the case where the frame represents the language *DL*

in which the possible situations are described this means that P is equivalent with the DL-sentences which correspond to the element b_P.

Have we then not reduced the counterfactual concept to a purely extensional one? Of course, whether this is so or not depends upon the language DL. Our original discussion of the decision process for counterfactuals may well have suggested that we indeed think of DL as purely extensional. But is this a reasonable assumption? In a narrow sense we might be asking whether or not DL contains such apparently intensional devices such as the counterfactual itself or other modal terms or constructions. If so, I believe it to be in keeping with the spirit of the present analysis to assume that DL does contain some such devices, but for the sake of argument we shall not make such an assumption, for there is still another sense in which DL may not be purely extensional. The basic predicates of L may contain certain intensional notions. For among these predicates there might be such predicates as "soluble" or "solid" which, according to some at least, are essentially counterfactual in meaning. To say that x is soluble, for example is to assert a battery of counterfactuals such as "If x were emerged it would dissolve", etc. I find it rather difficult to assess this theory of dispositional predicates. It might be argued for instance that the meaning of such predicates is bound up with the fundamental nature of substance. In which case such counterfactuals as "If a were emerged it would dissolve" would be *derivable* from statements such as "a is soluble". Let us, however, assume for the moment that DL must contain some such predicates. The result of the theorem might still appear rather surprising. For it claims that all counterfactuals are eliminable except those which are concealed beneath the facades of some of the primitive predicates of the description language.

There are, however, other reasons why this reduction of counterfactuals to complicated extensional constructions is largely illusory. In the first place the reduction depends on a particular choice of the relation $\supseteq$. This relation however may well be said to incorporate much, if not all, that has always been considered intractable in counterfactual constructions. What for example makes a situation in which I strike the match and in which there *is just as much oxygen as there* is in the actual world nearer to the actual world than a situation in which I strike the match and in which there is insufficient oxygen for ignition? This, you might say, is Goodman's cotenability problem all over again. This is, of course, a criticism which affects not only this analysis but those of Lewis and Stalnaker. Indeed, I do not

think such 'circularity' can be avoided in any analysis at such level of generality as the present one. I believe it possible to say something more definite about the relation $\supseteq$ within the narrower context of particular languages in which the counterfactual construction interacts with other than just truth-functional operators. But this is a topic for another occasion.

A second intensional component in our analysis might arise by consideration of the Boolean relation $\leqslant$. So far we have said next to nothing about these relations. When b_1 and b_2 represent (equivalence classes of) sentences of DL then $b_1 \leqslant b_2$ is to mean b_1 entails b_2. But "entails" in what sense precisely? The only notion of entailment which is both relevant and which we can clearly articulate is DL's notion of logical consequence. However, it is by no means clear that this is the correct notion of "entails". Indeed there seems good reason to include in the extension of $\leqslant$ pairs of the form $\langle b_1, b_2 \rangle$ where b_2 follows from b_1 with the help of certain natural laws. For surely when we consider the various alternative situations in which a given hypothesis could have been true we surely leave out all those alternatives which are physically impossible, and equally we ignore when contemplating how further details of an already partially described situation could be filled in, those additional specifications of detail which would render the existing specifications physically (though not logically) incoherent. If we opt for such an interpretation of $\leqslant$ we thereby introduce a further intensional component into our analysis.

We thus find ourselves in the following situation. Once we settle on a particular interpretation of $\supseteq$ and $\geqslant$ we thereby assent, according to my theory, to an equivalence between counterfactual conditionals and corresponding constructions which contain only extensional operators. This still does not automatically make the counterfactual conditional into an extensional construction. In the first place, it is plausible that the exact extensions of $\supseteq$ and $\leqslant$ are not fully determined by the meaning of the counterfactual conditional. If this is indeed so then it will be impossible to pinpoint any particular extensional sentence to which a given counterfactual is equivalent. Secondly, the question for the correct interpretation of $\leqslant$ and $\supseteq$ is complicated by its apparently presupposing a previous understanding of certain counterfactuals or cognate constructions such as law-like generalisations. It is an interesting problem whether something more specific could be said concerning which counterfactuals should be fed into the characterisations of $\supseteq$ and $\geqslant$. Given that this impact has been spelled

out we would then be faced with the potentially quite interesting problem of whether the analysis of counterfactuals so produced is coherent: is there a function T from conditional sentences to extensional sentences so that when we determine the relations $\trianglerighteq$ and $\geqslant$ in terms of the T-counterparts relevant to those relations then the analysis determines as the extensional equivalents of counterfactuals precisely their values under T.

We have here an example of a construction which would be extensional on the assumption that all the components of our theory are fully defined but whose intensionality resides precisely in that not all the components are fully defined, so that a number of different non-equivalent, extensional sentences may be different paraphrases for the same sentence in which the construction figures.

6. COMPARATIVE PLAUSIBILITY AND COMPARATIVE SIMILARITY

As stated at the outset of our discussion we have opted to take the relation of comparative plausibility on partial descriptions of possible worlds as primitive; the relation of comparative similarity on possible worlds is then taken as the derived relation. But how exactly are we to define it? It will be helpful at this point to recall that corresponding to every possible world there is a unique (with respect to a given Boolean frame) approximation sequence which generates it; the successive members of this sequence may be regarded as representing ever more detailed descriptions of the possible worlds. With this in mind the following definition seems correct:

$$u \overset{w}{\lesssim} v \leftrightarrows_{\text{def}} (\forall k)(\exists m, n \geqslant k)(u_n \overset{w}{\trianglerighteq} v_m).$$

The content of the definition is reasonably easy to grasp: for every degree of detail chosen we can find partial descriptions of u and v, of at least the chosen degree of detail, which satisfy the original relation of relative plausibility. The usual definition of the strict $\sim (v \overset{w}{\lesssim} u)$ leads to the following equivalence:

$$u \overset{w}{\lesssim} v \leftrightarrows (\exists k)(\forall m, n \geqslant k)(u_m \overset{w}{\trianglerighteq} v_n).$$

This may be reason for some concern. Under this definition to establish that $u \overset{w}{\lesssim} v$ it is sufficient to establish that for all finite levels of description u is closer to w than v. But surely it is possible for one world to be closer than a

second (to the actual world), at all finite levels of description, and yet not be so in the limit. I believe this objection would be more telling if our notion of comparative similarity was, like that of Lewis, a purely metaphysical one; but ours is clearly not. Our development has been motivated be a desire to come closer to the way we evaluate counterfactuals; and it is exactly this constraint on our theory which demands that decisions concerning comparative similarity be based on finite amounts of information about the worlds in question.

THEOREM 6.1. Let $u, v, z\ w \in U(B)$ for some Boolean frame B. The derived relation $\geqslant$ satisfies:

(L1) $\quad u \overset{w}{\gtrsim} z \,\&\, z \overset{w}{\gtrsim} v \Rightarrow u \overset{w}{\gtrsim} v$

(L2) $\quad u \overset{w}{\gtrsim} v \vee v \overset{w}{\gtrsim} u$

(L3) $\quad \sim (u = v) \Rightarrow u \overset{u}{\gtrsim} v.$

Proof. Assume $u \overset{w}{\gtrsim} z$ and $z \overset{w}{\gtrsim} v$. Choose some k. By $u \overset{w}{\gtrsim} z$ there exist m, $n \geqslant k$ with $u_n \overset{w}{\supseteq} z_m$. Choose $k' \geqslant m, n$. By $z \overset{w}{\gtrsim} v$ there exist $r, m' > k'$ such that $z_{m'} \overset{w}{\supseteq} v_r$. Now since $z_{m'} \leqslant z_m (m' \geqslant m)$ it follows from (B5) that $u_n \overset{w}{\supseteq} z_{m'}$ (as $u_n \overset{w}{\supseteq} z_m$). By the transitivity of $\overset{w}{\supseteq}$ we have $u_n \overset{w}{\supseteq} v_r$ and so by definition $u \overset{w}{\gtrsim} v$. So $\overset{w}{\gtrsim}$ is transitive. For (L2) assume $\sim (u \overset{w}{\gtrsim} v)$. By definition $(\exists k)(\forall m, n \geqslant k)(\sim (u_n \overset{w}{\supseteq} v_m))$. By linearity of $\overset{w}{\supseteq}$ we have $(\exists k)(\forall m, n \geqslant k)(v_m \overset{w}{\supseteq} u_n)$. But then obviously, $v \overset{w}{\gtrsim} u$ – for every k' chosen ensure the m, n required are larger than the k for which $v_m \overset{w}{\supseteq} u_n$, for $m, n \geqslant k$. For (L3) assume $\sim (u = v)$. Let k be large enough so that $u_k \cap v_k = 0$. Since for each $m, n \geqslant k, u_m \cap v_n = 0$ we have by (B3) $u_m \overset{u}{\supseteq} v_n$. Hence $u \overset{v}{\gtrsim} v$.

These properties are exactly those which are deemed necessary by Lewis for any such relation of comparative similarity. We shall say more about such relations a little later.

We can thus characterise the relation of relative similarity in terms of the relation of relative plausibility but can we characterise the latter in terms of the former? According to our informal discussion we were inclined to assert that a is at least as plausible as b, exactly when we can match any world v, of which b is a correct description, by a world in which a is a

correct description and which is at least as similar as v. The following result guarantees that this intuition is preserved in our theory.

THEOREM 6.2. For each $a, b \in B$

$$a \overset{w}{\supseteq} b \Leftrightarrow (\forall v \supseteq b)(\exists u \supseteq a)(u \overset{w}{>} v).$$

Proof. The cases where $b = 0$ and $a = 0$ are straightforward – use (B7) and (B8). So assume $a \neq 0$ and $b \neq 0$. Assume $a \overset{w}{\supseteq} b$. Let $v \supseteq b$. Choose $k \geqslant d(a), d(b)$. By (B5) $a \overset{w}{\supseteq} v_k$ since $v_k \leqslant b$. By (B6) we can find an atom $u_k \leqslant a$ of B_k such that $u_k \overset{w}{\supseteq} v_k$. To see this notice that $a = u_1 \cup \ldots \cup u_n$ for atoms $u_k \in B_k$. Applying (B5) and (B6), once again, we can find an atom u_{k+1} of B_{k+1}, $u_{k+1} \leqslant u_k$ such that $u_{k+1} \overset{w}{\supseteq} v_{k+1}$. Proceeding in this way we can construct a sequence $u_k \geqslant u_{k+1} \geqslant u_{k+2} > \ldots$ such that $u_{k+i} \overset{w}{\supseteq} v_{k+i}$, $i \geqslant 0$. Let $u \in u_{k+i}$, $\forall i \geqslant 0$. Clearly $u \overset{w}{>} v$. Conversely assume $\sim (a \overset{w}{\supseteq} b)$. Let $k \geqslant d(a), d(b)$. By (B5) there exists $v_k \leqslant b$, v_k an atom in B_k, such that $\sim (a \overset{w}{\supseteq} v_k)$. By repeated application of (B5) we can construct a sequence $v_k \geqslant v_{k+1} \geqslant v_{k+2} \geqslant \ldots$ such that v_{k+i} is an atom in B_{k+i} and $\sim (a \overset{w}{\supseteq} v_{k+i})$ for $i \geqslant 0$. Let $v \supseteq v_{k+i}$, $i \geqslant 0$. By (B4) we have for each $u \supseteq a$ and each $m \geqslant k$, $i \geqslant 0$, $\sim (u_m \overset{w}{\supseteq} v_{k+i})$. It follows that for each $u \supseteq a$ there exists a k such that $(\forall m, n \geqslant k)(\sim (u_m \overset{w}{\supseteq} v_n))$; hence for each $u \supseteq a$, $\sim (u \overset{w}{\supseteq} v)$. This implies that

$$(\exists v \supseteq b)(\forall u \supseteq a)(v \overset{w}{>} u).$$

This state of affairs is quite satisfactory. It leads one to believe that the formalization chosen has captured the appropriate intuitions. The relationship elaborated in the theorem is the relationship which guided our intuitions in the development of principle (B8); if you recall (B8) emerged as the outcome of our informal investigation into the relationship between the relations of comparative plausibility and comparative similarity. The formal theory, which is the result of these informal insights, has exactly this relationship as a formal consequence.

We have to take the first step towards a comparison between our theory and that based on comparative similarity. We ought now to go further and relate the truth-conditions. To do this we need some notions from the writings of D. Lewis. A non-empty set $U \subseteq U(B)$, is called a *w-sphere* just in case $u \in U$ and $v \overset{w}{>} u$ implies $v \in U$. Lewis has shown that the totality of *w-spheres* forms a nested system of sets with the singleton set

$\{w\}$ at the centre; the smaller the sphere containing the world the more similar the world is to w.

THEOREM 6.3. Let M be a Boolean Model and let $[P] = \{w \in U(B): f(P) \in \omega\}$. Then the following hold:

(A1) $[P \,\&\, Q] = [P] \cap [Q]$

(A2) $[\sim P] = U(B) - [P]$

(A3) $[P \,\Box\!\!\rightarrow Q] = \{w \in U(B)$: either $[P]$ is empty or $U \cap [P] \subseteq [Q]$ for some w-sphere with $U \cap [P]$ non-empty$\}$.

Proof. Clause (A3) is the only non-trivial case; the rest follow directly from the obvious properties of f: $f(P \,\&\, Q) = f(P) \cap f(Q)$ and $f(\sim P) = f(P)^*$. So we concentrate on (A3). The vacuous case is easy: $f(P) = 0$ just in case $[P]$ is empty. More generally assume $w \in [P \,\Box\!\!\rightarrow Q]$. By definition $f(P \,\Box\!\!\rightarrow Q) \in w$ and so there exists an n such that $w_n \leqslant f(P \,\Box\!\!\rightarrow Q)$. Hence, there exists an n (choose $n \geqslant d(P \,\Box\!\!\rightarrow Q)$) such that $w_n \models P \,\Box\!\!\rightarrow Q$. Hence, there exists a $\langle w_n, P \rangle$-decision tree p such that for each $c \in N_p(w_n)$, $\hat{c} \models Q$. In fact we can ensure that the tree is decorated with atoms T_r where $r \geqslant d(Q)$; hence $\hat{c} \models Q$ iff $\hat{c} \leqslant f(Q)$ for each branch $c \in N_p(w_n)$. Furthermore (see last section for notation)

$$\overline{N_p(w_n)} \overset{w_n}{\supset} \overline{B(p) - N_p(w_n)};$$

and so by (B10)

$$\overline{N_p(w_n)} \overset{w}{\supset} \overline{B(p) - N_p(w_n)}.$$

By the relationship between $\overset{w}{\supset}$ and $\overset{w}{\geq} - (a \overset{w}{\supset} b \leftrightarrow (\exists u \ni a)(\forall v \ni b)(u \overset{w}{\geq} v))$ there is some $u \ni \overline{N_p(w_n)}$ such that $u \overset{w}{\geq} v$ for each $v \ni \overline{B(p) - N_p(w_n)}$. Clearly $U = \{v \ni U(B)$: $v \overset{w}{\geq} u\}$ is a w-sphere. Claim $U \cap [P] \subseteq \{v$: $\overline{N_p(w_n)} \ni v\}$ – otherwise there is $v \in U \cap [P]$ and $\sim (v \ni \overline{N_p(w_n)})$ and so $v \overset{w}{\geq} u$ and $u \overset{w}{\geq} v$ which is a contradiction. Furthermore, for each $c \in N_p(w_n)$, $\hat{c} \leqslant f(Q)$ and so $\{u: \overline{N_p(w_n)} \ni u\} \subseteq [Q]$. It follows that $U \cap [P] \subseteq [Q]$.

Conversely, suppose $U \cap [P] \subseteq [Q]$ for some w-sphere U. Since $U \cap [P] \subseteq [Q] \cap [P]$ and $[P] \cap [\sim Q] \subseteq ([P] - U)$ it follows that $f(P) \cap f(Q) \overset{w}{\supset} f(P) \cap f(\sim P)$. This follows because U is a w-sphere and so

for some $u \in U$, $u \overset{w}{\succsim} v$ for each $v \in [P] - U$; and hence the claim follows by the relationship between $\overset{w}{\supset}$ and $\overset{w}{\succsim}$ and the definition of [ ].

Build a P-decision tree p as follows. Put $f(P)$ at the vertex and as ancestors place all the atoms t in T_m with $t \leqslant f(P)$ and where $m \geqslant d(P)$, $d(Q)$. Choose n large enough so that for each pair $s, t \in B(p)$ either $s \overset{w_n}{\supset} t$ or $t \overset{w_n}{\supset} s$ or $s \overset{w_n}{\equiv} t$; and (by (B10) large enough so that $f(P) \cap f(Q) \overset{w_n}{\supset} f(P) \cap f(\sim Q)$. Now suppose $t \in N_p(w_n)$. Then $t \leqslant f(P) \cap f(Q)$ for otherwise $t \leqslant f(P) \cap f(\sim Q)$. But there is a $u \ni f(P) \cap f(Q)$ with $u \ni s$, for some $s \in B(p)$ and $u \overset{w}{\succsim} v$ for each $v \ni t$. This contradicts $t \in N_p(w_n)$. But then each $t \in N_p(w_n)$ satisfies $t \leqslant f(Q)$ and, hence, because of the choice of m, $t \models Q$.

The proof of this result is quite interesting in its own right. It displays a correspondence between decision trees and spheres of possible worlds. The more you refine the decision tree the smaller the corresponding sphere. This seems intuitively correct: the more information you have about the alternative worlds the smaller the sphere of possible worlds under scrutiny.

7. POSSIBLE WORLDS AND COMPARATIVE SIMILARITY

You may find all this rather unconvincing; you may along with Lewis believe that there are possible worlds other than the one we inhabit and moreover that a relation of similarity between such alternative worlds provides the correct analysis of counterfactuals. Rather than enter into this dispute I will instead discuss some relationships between the two theories. In any case, I have said enough about why I believe the Lewis theory to be unsatisfactory.

We shall show that the two notions of validity coincide: for every sentence P of $L_{\Box\rightarrow}$, P is valid with respect to all Boolean models exactly when it is valid with respect to all Lewis models. To justify this remark we first remind the reader of the basic framework of Lewis. We shall use the formulation based upon his relations of comparative similarity and introduce the notion of a nested system of spheres as a derived concept. Lewis has taught us that the two formulations are entirely equivalent.

For possible worlds u, v, w we introduce the notation $u \overset{w}{\succsim} v$ to mean u is at least as similar as v to w. A structure $L = \langle W, \leqslant \rangle$, where W is a non-empty

set (of possible worlds) and $\leqslant$ is a relation (of comparative similarity) which satisfies (L1), (L2), (L3) we call a *Lewis frame*.

How exactly are we to recover the structure of a Boolean frame from that of a Lewis frame?

According to our original intuitions we should regard a partial description a as at least as plausible as a partial description b just in case for any possible world v in b there exists a possible world u in a which is at least as similar as v to the actual world. I take it that propositions (or sets of possible worlds) are, in this context at least, to play the role of partial descriptions. The above then suggest the following definition:

$$a \overset{w}{\geq} b \Leftrightarrow_{\text{def}} (\forall v \in b)(\exists u \in a)(u \overset{w}{>} v).$$

We shall use a, b, c etc. to refer to propositions.

THEOREM 7.1. The relation $\overset{w}{\geq}$ so defined satisfies the axioms (B1)–(B7) of a Boolean frame.

Proof. For (B1) assume $a \overset{w}{\geq} b$ and $b \overset{w}{\geq} c$. Then for each $z \in c$ we can find $v \in b$ such that $v \overset{w}{>} z$. But we can find $u \in a$ such that $u \overset{w}{>} v$ so by (L1) $u \overset{w}{>} z$.

In (B2) assume $\sim (a \overset{w}{\geq} b)$. By definition and (L2) there exists $u \in b$ such that $u \overset{w}{<} v$ for each $v \in a$. It follows that $b \overset{w}{\geq} a$. For (B3) we have to show that $w \in b$ and $a \cap b = 0$ implies $b \overset{w}{>} a$, i.e., $(\exists u \in b)(\forall v \in b)(u \overset{w}{>} v)$. We choose $u = w$. Since $w \in b$ and $a \cap b = 0$ it follows that $\sim (w = v)$ for each $v \in b$. By (L3) $w \overset{w}{>} v$ for each $v \in b$. (B4) follows immediately since $v \in a$ implies $u \in a \cup a'$ – hence $a \overset{w}{\geq} b$ implies $a \cup a' \overset{w}{\geq} b$. (B5) from right to left is also trivial since $v \in b$ implies $v \in b \cup b'$. Conversely, $v \in b \cup b'$ implies $v \in b$ or $v \in b'$ and so (B5), from right to left, follows. Axiom (B7) is trivially true.

The closest Lewis comes to our relation of relative plausibility is the relation he refers to as a relation of 'comparative possibility'. We shall follow Lewis and write $a \gtrsim_w b$ to mean that the proposition a is at least as possible, at the world w, as the proposition b. The strict relation is defined in the normal way by

$$a \gtrsim_w b \Leftrightarrow \sim (b \gtrsim_w a).$$

These relations are meant to satisfy the following five conditions.

(C1) $\quad a \gtrsim_w b \;\&\; b \gtrsim_w c \Rightarrow a \gtrsim_w c$

(C2) $a \gtrsim_w b \vee b \gtrsim_w a$

(C3) $(\forall b \in J)(a \gtrsim_w b) \Leftrightarrow a \gtrsim_w \cup J$

(C4) $w \in b \Leftrightarrow (\forall a)(b \gtrsim_w a)$

(C5) $(\forall b \in J)(\{u\} \gtrsim_w b) \Rightarrow \{u\} \gtrsim_w \cup J.$

Here J is a set of propositions and $\cup J = \cup \{b: b \in J\}$.

We know from Lewis that axioms (C1)–(C5) are equivalent to (L1)–(L3). Notice also that axioms (B1)–(B3), restricted to singleton propositions, are just (L1)–(L3). It follows that, where propositions are to be represented as sets of possible worlds, the three axiom sets (L1)–(L3), (B1)–(B7), (C1)–(C5) are equivalent. Of course, in contexts where propositions are to be thought of as partial descriptions of possible worlds, certain of the axioms in (C1)–(C5) are rather obscure. What, for example, are we to make of (C3) in such contexts?

Despite all these obvious equivalences the relations $\overset{w}{\geq}$ do not satisfy *all* the axioms of a Boolean frame; (B9) and (B10) are not even obviously expressible in such a general context. These axioms crucially rely on propositions as being partial descriptions rather than sets of possible worlds. To obtain a structure which has these additional properties we need to be more devious in our construction. We need to use the notion of a Lewis model rather than just the idea of a Lewis frame.

A *Lewis model* N consists of a Lewis frame $L = \langle W, \leqslant \rangle$ together with a function $[\ ]: L_{\Box\rightarrow} \rightarrow P(W)$ (the power set of W) which satisfies (A1), (A2) and (A3).

To obtain a Boolean frame from a Lewis model N we proceed much the same as before. The major change concerns the actual Boolean algebra. Instead of using the full power set of W we select just those propositions which can be expressed in $L_{\Box\rightarrow}$. More exactly, let $B = \{[P]: P \in L_{\Box\rightarrow}\}$ and the operations of the algebra be the usual set-theoretic ones. Since B is denumerable we know from a previous result that B is constructive with some measure d. The ultrafilters of B do not necessarily exhaust the ultrafilters in the full power set algebra $P(W)$. Furthermore, we have $a \overset{w}{\geq} b$ defined only for $w \in W$; we need to extend the definition to $U(B)$ – ultrafilters of B. Presumably, $a \overset{X}{\geq} b$ is true just in case $a \overset{w}{\geq} b$ holds for each possible world w in X:

$$a \overset{X}{\trianglerighteq} b \Leftrightarrow_{\text{def}} (\forall w \in \hat{X})(a \overset{w}{\trianglerighteq} c)$$

where $\hat{X} = \cap \{[P]\colon [P] \in X\}$ and $a, b \in B$.

THEOREM 7.2. The structure $B = \langle B, \trianglerighteq, d \rangle$, so defined, satisfies the axioms of a Boolean frame.

Proof. (B1), (B4) and (B5) follow from the corresponding properties for $\overset{w}{\trianglerighteq}$. For the rest we need a couple of preliminary observations. Lewis has shown that there is some sentence '$a \trianglerighteq b$' of $L_{\Box\rightarrow}$ such that

$$w \in [a \trianglerighteq b] \quad \text{iff } a \overset{w}{\trianglerighteq} b$$

(in fact $a \trianglerighteq b$ is just $(a \check{\vee} b \Box\rightarrow a \check{\vee} b) \supset (a \check{\vee} b \Box\rightarrow a)$. The antecedent of the material conditional is there to allow for impossible antecedents). Next observe that if $\hat{X} \cap [P]$ is non-empty then $\hat{X} \subseteq [P]$ – this much is clear since X is an ultrafilter in B and so X contains $[P]$ or $[\sim P]$ but not both. But then $(\exists w \in \hat{X})(a \overset{w}{\trianglerighteq} b) \Leftrightarrow (\exists w \in \hat{X})(w \in [a \trianglerighteq b]) \Leftrightarrow \hat{X} \cap [a \trianglerighteq b]$ non-empty $\Leftrightarrow \hat{X} \subseteq [a \trianglerighteq b] \Leftrightarrow (\forall w \in \hat{X})(a \overset{w}{\trianglerighteq} b)$. The same applies to the strict relation: $(\exists w \in \hat{X})(b \overset{w}{\triangleright} a) \Leftrightarrow (\forall w \in \hat{x})(b \overset{w}{\triangleright} a))$ where $b \overset{w}{\triangleright} a \Leftrightarrow_{\text{def}} \sim (a \overset{w}{\trianglerighteq} b)$.

We return to the audit of our properties. (B2) follows immediately given the above observation. For (B3) suppose $a \cap b = 0$ in B and $a \in X$. Then presumably $a \cap b = 0$ in $P(W)$. Moreover, $u \in \hat{X}$ implies $u \in a$ and so $\sim (u = v)$ for each $v \in b$; and so by (B3) for $\overset{u}{\triangleright}$ we have $a \overset{u}{\triangleright} b$. But then we have $(\exists u \in \hat{X})(a \overset{u}{\triangleright} b)$. Which is sufficient to guarantee $a \overset{X}{\triangleright} b$. Next, we establish (B9) ((B10) is similar). Notice that $a \overset{X}{\trianglerighteq} b \Leftrightarrow (\forall w \in \hat{X})(a \overset{w}{\trianglerighteq} b) \Leftrightarrow \hat{X} \subseteq [a \trianglerighteq b]$. But the latter is equivalent to $[a \trianglerighteq b] \in X$. Let $X_1, X_2, \ldots$ be the *uf*-sequence of X; then there exists k such that $X_k \subseteq [a \trianglerighteq b]$, i.e.,

$$[a \trianglerighteq b] \in X \Leftrightarrow (\exists k)(X_k \subseteq [a \trianglerighteq b])$$

(the measure is obtained from some enumeration of B). Hence $a \overset{X}{\trianglerighteq} b \Leftrightarrow (\exists k)(X_k \subseteq [a \trianglerighteq b])$; which is equivalent to $(\exists k)(\forall w \in X_k)(a \overset{w}{\trianglerighteq} b)$, i.e., $(\exists k)(a \overset{X_k}{\trianglerighteq} b)$.

For purposes of identification we refer to this derived Boolean frame as B_N where N is the Lewis model from whence it came. In fact we can obtain a Boolean *model* from N, $B_N = \langle B_N, f \rangle$ where $f\colon L \to B$ is just $f(P) = [P]$. Our next result informs us that the two models are equivalent. The proof is

almost identical to the corresponding proof in the previous section and so we omit it.

THEOREM 7.3. Let N be a Lewis model and M_N the corresponding Boolean model. Then for each $P \in L_{\Box\rightarrow}$, $\{w: f(P) \in w\} = [P]$.

What information does this result give us about the underlying notions of validity? Let Val_B be the set of all sentences P in L such that for each Boolean model $f(P) = 1$; and let Val_L be the set of P such that for each Lewis model $[P] = W$. This result, together with that of the previous section, gives us the $\mathrm{Val}_L = \mathrm{Val}_B$; or in words, validity in Lewis models corresponds exactly to validity in Boolean models. The equivalence of the two theories allows one to choose the theory which one finds the philosophically/semantically more acceptable without affecting the underlying concept of validity.

The theory based on Boolean models is clearly motivated by more pragmatic considerations than the account of Lewis. It tries to relate to the way we might actually evaluate counterfactuals whereas Lewis is motivated by more metaphical considerations.

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NOTES

[1] It has been observed in the meantime, in particular by Stalnaker himself, that the difference between Lewis' theory and his own account becomes much narrower if one allows that the function f may not always be fully determined by the context of utterance. It would seem that in case $f(S, \phi)$ is not determined (1) would automatically be without truth-value. However, in certain such cases it is still possible to attribute a truth-value to (1), viz. when for all functions f compatible with the context, ψ has the same truth-value. If, on the other hand, ψ is true in some of these while false in some others then both (1) and (3) would be regarded as truth-valueless. This still does not seem fully satisfactory – in some cases of this kind (3), say, would intuitively seem false rather than undetermined, and (2) true – c.f. [30].

[2] Here and hereafter $\alpha \Box\!\!\rightarrow \beta$ stands for 'if it were (had been) the case that α then it would be (have been) the case that β'. In this paper I do not distinguish between the 'were' and 'had been' constructions.

[3] We must stress once again that this is not offered as a total solution to Goodman's problem; I do not believe that this can be done in such a general framework as this paper presupposes.

[4] Strictly speaking, a disjunction of such assertions; but I shall ignore this complication.

[5] This paraphrase of s is called the distributive normal form.

[6] For a more detailed account of these matters we refer the reader to Hintikka's own work.

[7] The reader who is unfamiliar with Boolean algebra is advised to consult Sikorski [1964]: *Boolean Algebras*, Springer.

[8] A *Filter* of a Boolean algebra B is a proper subset F of B such that (i) $b \in F$ and $b < b'$ implies $b' \in F$; (ii) $b, b' \in F$ implies $b \wedge b' \in F$. An *Ultrafilter* of $\mathscr{B}$ is a filter such that for every $b \in \mathscr{B}$ either $b \in F$ or $b^* \in F$. We denote the ultrafilters of $\mathscr{B}$ by $\mathscr{U}(\mathscr{B})$.

[9] This is, of course, made acceptable by the Stone representation theorem.

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